An Energy Balance Model for Arctic Sea Ice

Kaitlin Hill

October 10, 2017

Outline

Motivation: Arctic sea ice decline

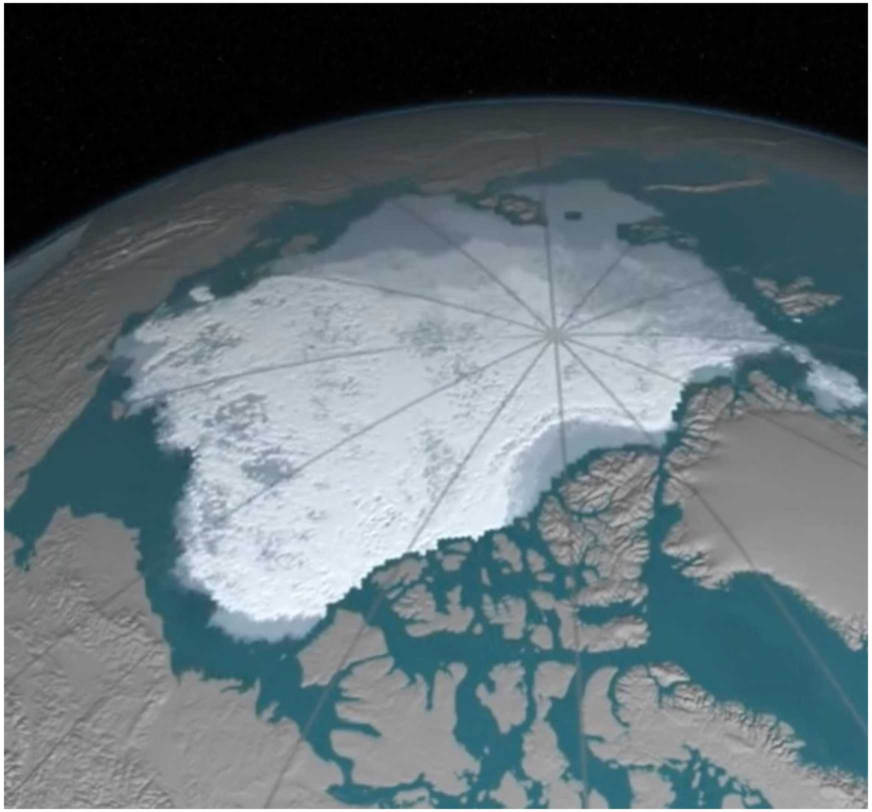
Energy balance model

Bifurcation analysis

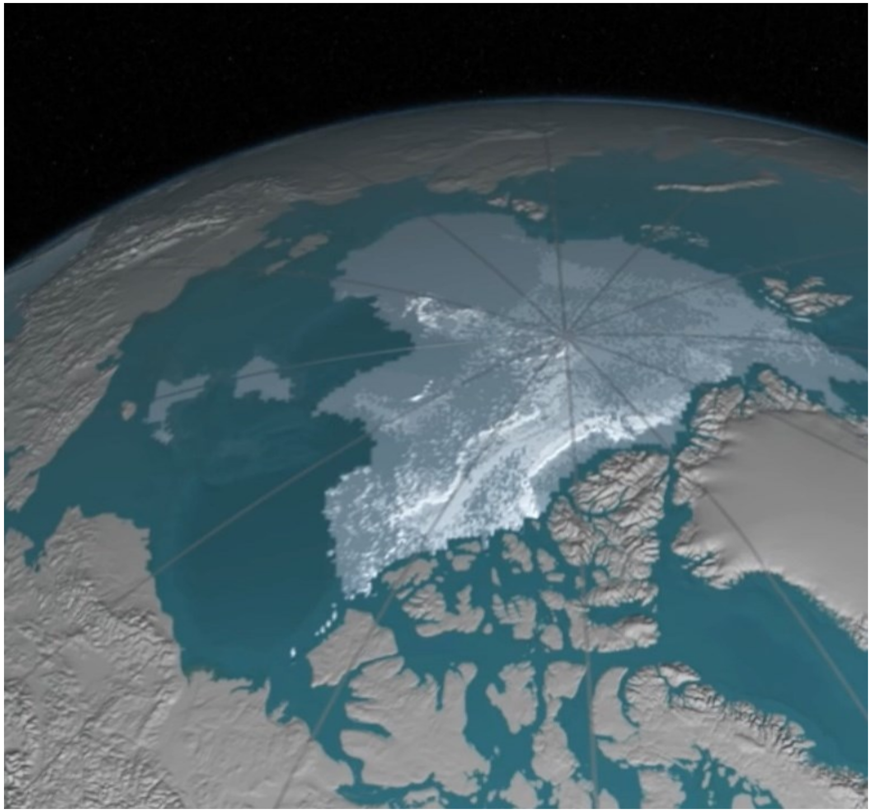
Results

Arctic sea ice age

Sept 1986

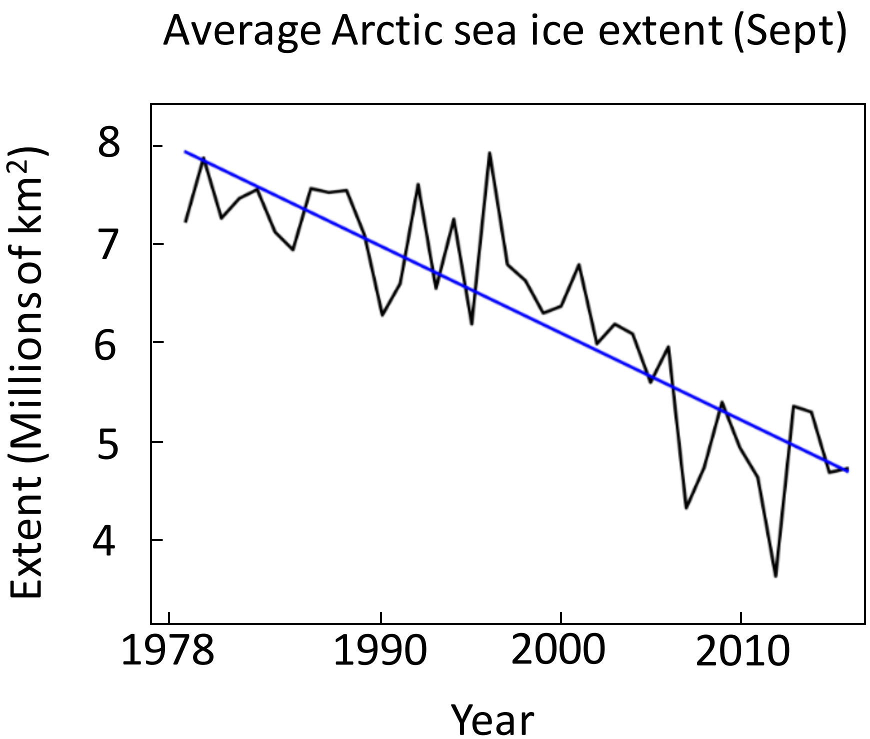


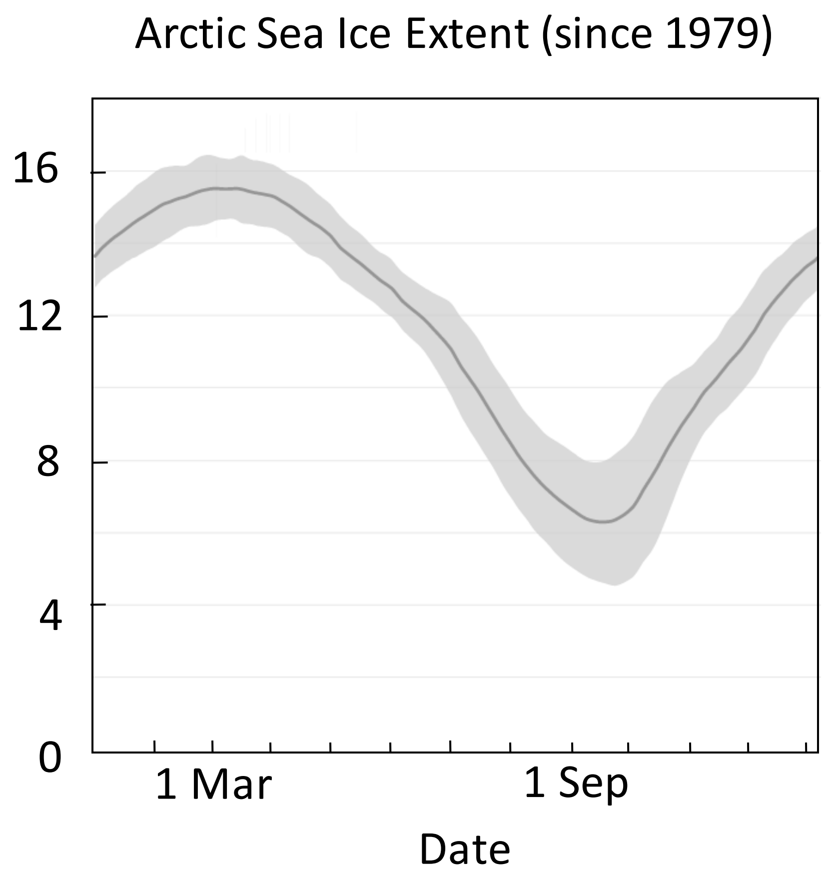
Sept 2016



(NASA Scientific Visualization Studio)

Recent decline in Arctic sea ice extent





(nsidc.org)

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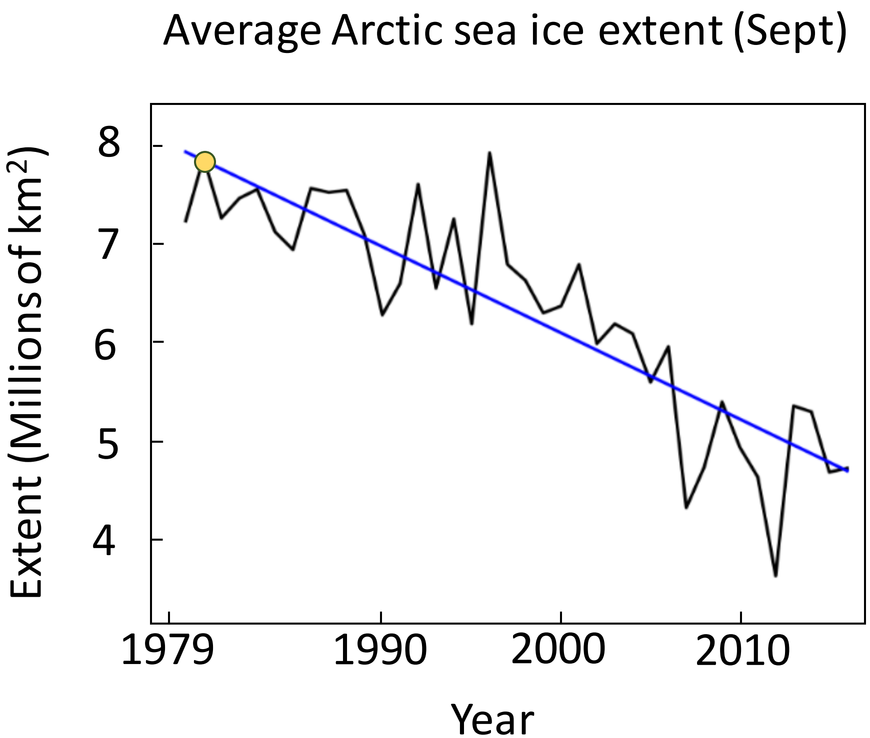

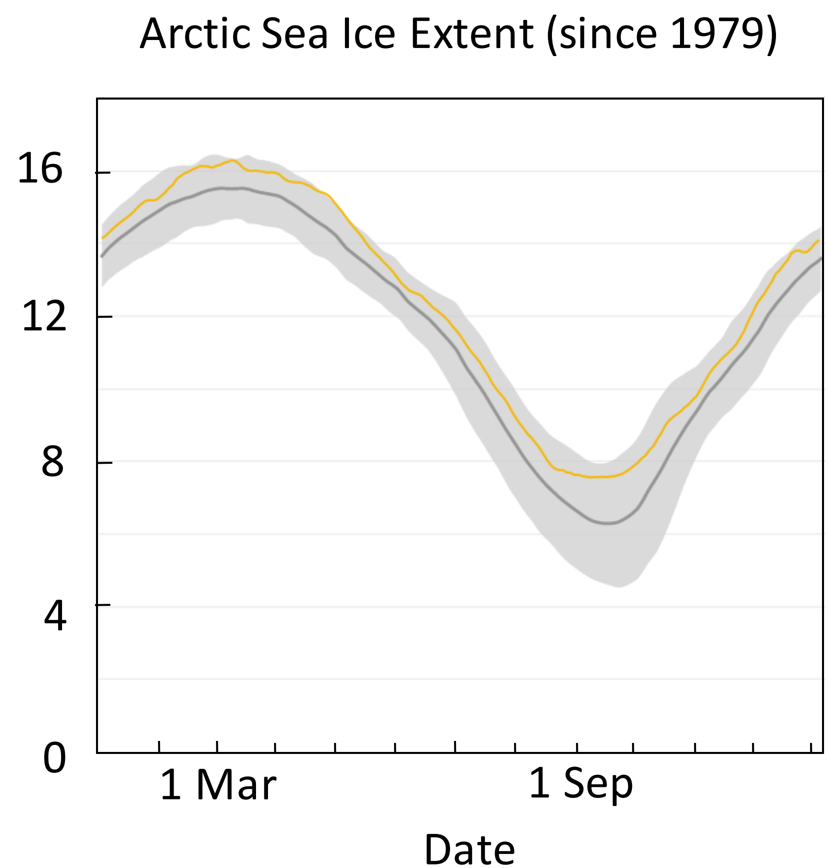

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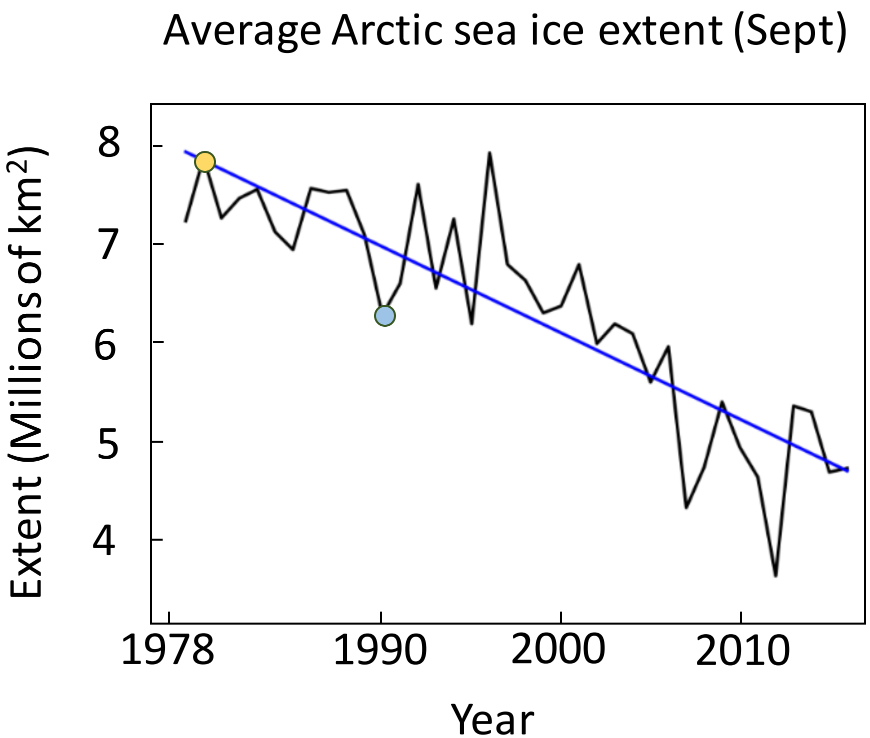

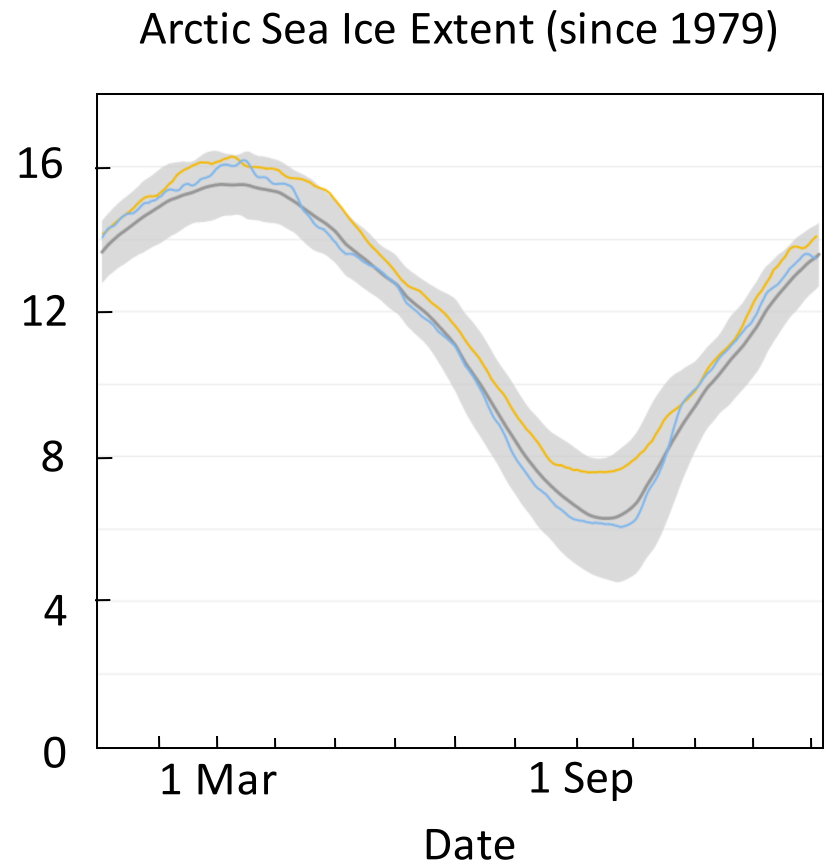

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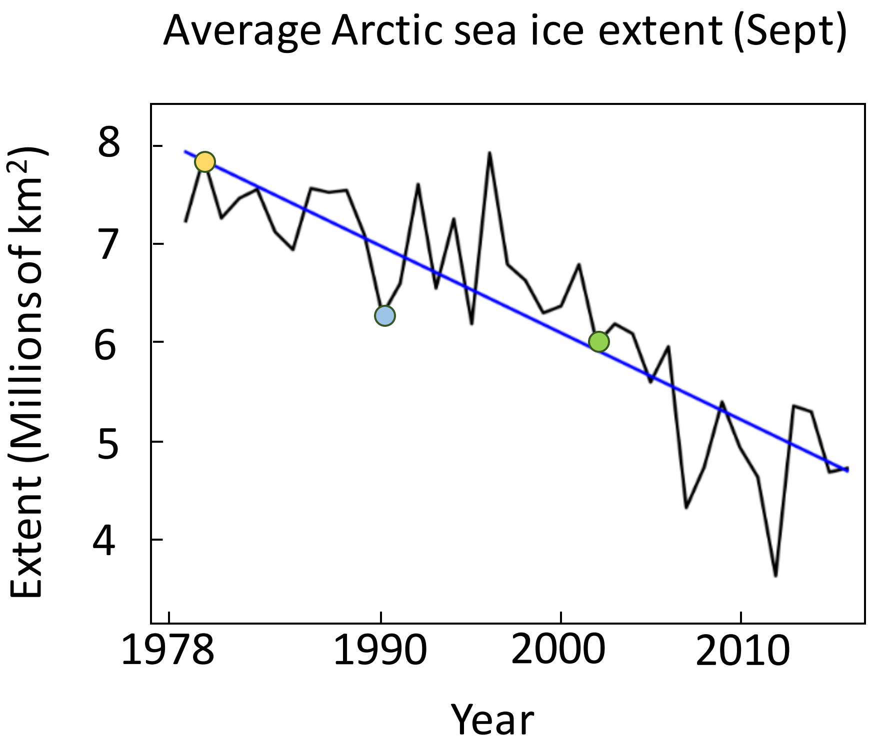

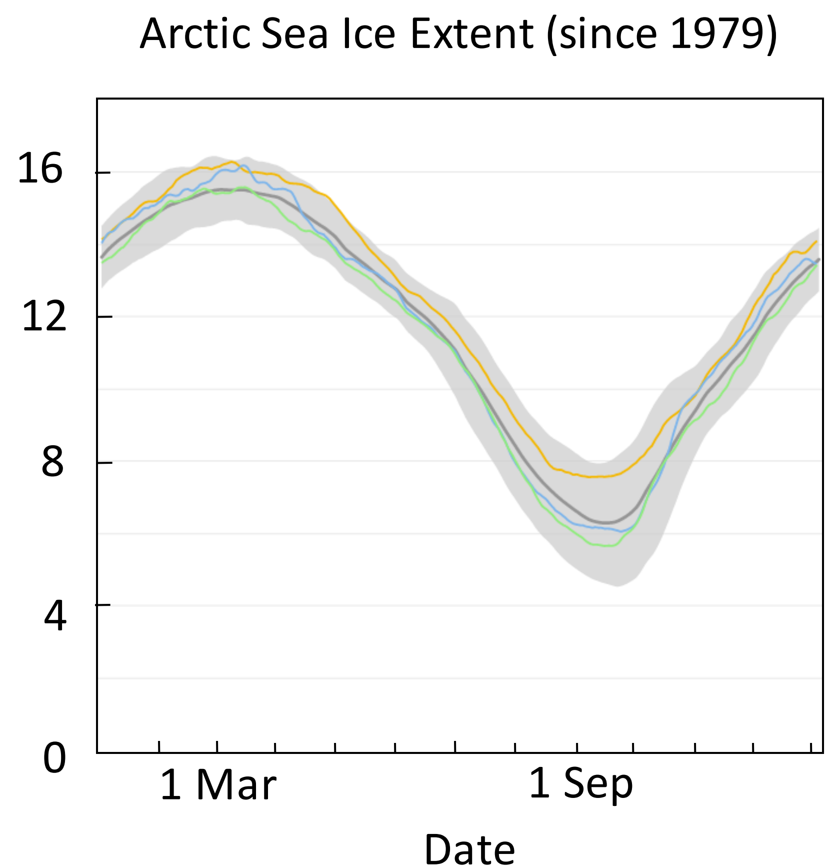

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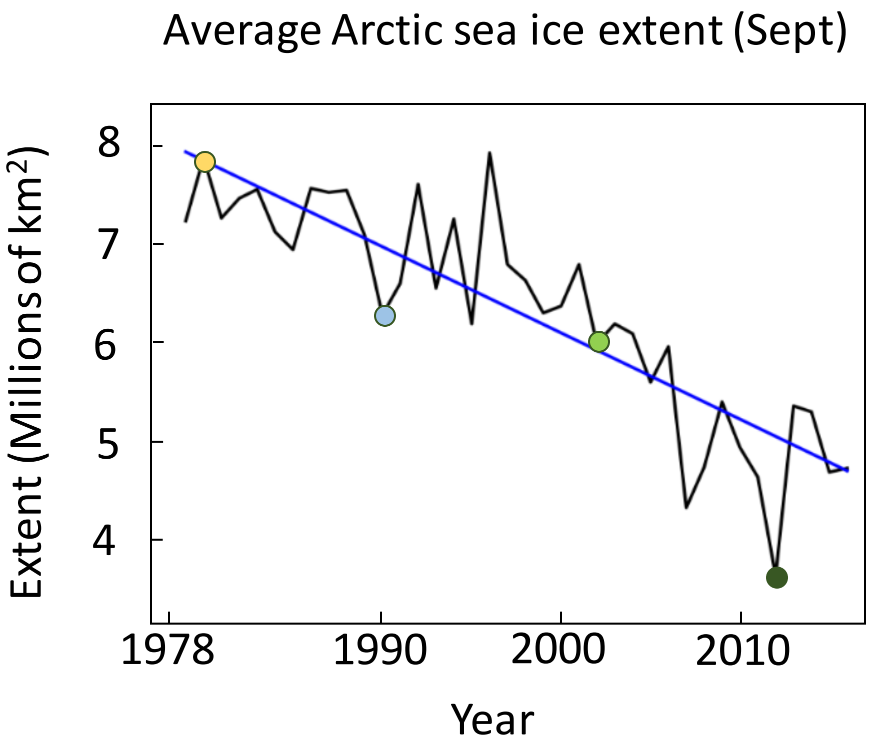
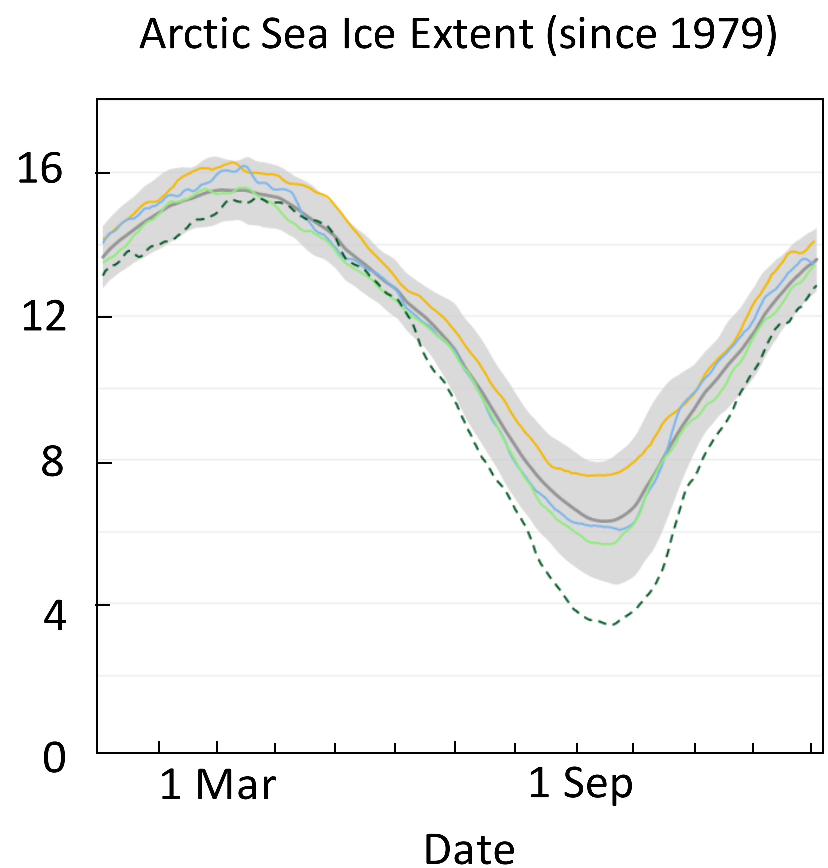




(nsidc.org)

Arctic energy balance model

$$\frac{dE}{dt} = \boxed{\big(1 - \alpha(E)\big)F_s(t)} - \boxed{(F_l(t) + BT(E,t))}$$

incoming outgoing

(Eisenman and Wettlaufer 2009)

Arctic energy balance model

$$\frac{dE}{dt} = \underbrace{\boxed{\big(1-\alpha(E)\big)F_s(t)}}_{\text{incoming}} - \underbrace{\boxed{(F_l(t) + BT(E,t))}}_{\text{outgoing}}$$

Major components:

- E has no spatial extent
- Seasonally-varying incoming solar radiation
- Ice-albedo feedback
- State-dependent definition of energy:

$$E \propto \begin{cases} \text{Ocean mixed−layer temperature } (E > 0) \\ \text{Ice thickness } (E \leq 0) \end{cases}$$

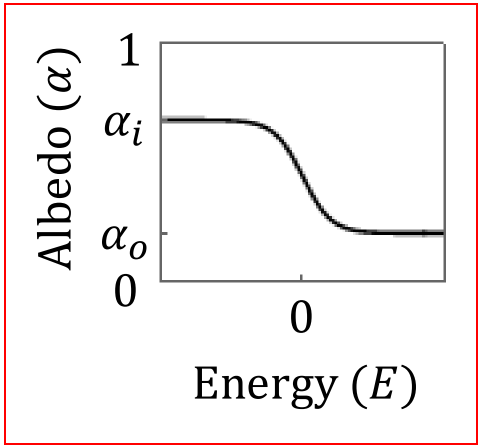
- Sea ice thermodynamics

(Eisenman and Wettlaufer 2009)

Arctic energy balance model

$$\frac{dE}{dt} = \big(1 - \underline{\alpha(E)}\big)F_s(t) \; - \; (F_l(t) + BT(E,t))$$

albedo

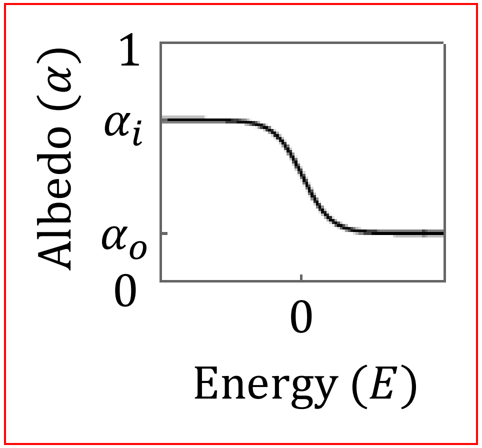


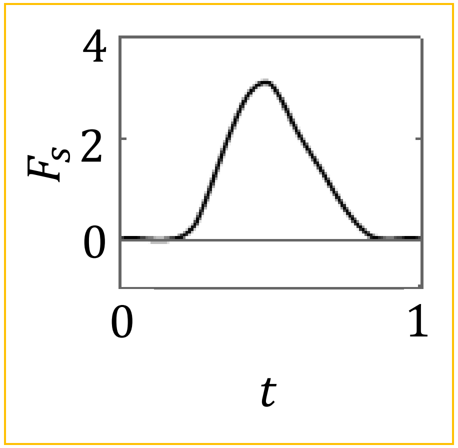
(Eisenman and Wettlaufer 2009)

Arctic energy balance model

$$\frac{dE}{dt} = \big(1 - \alpha(E)\big)F_s(t) - \left(F_l(t) + BT(E,t)\right)$$

albedo incoming solar radiation



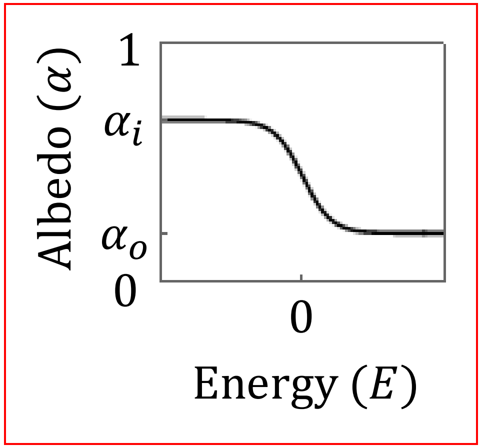


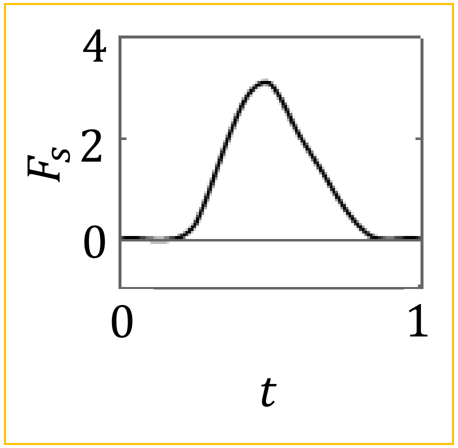
(Eisenman and Wettlaufer 2009)

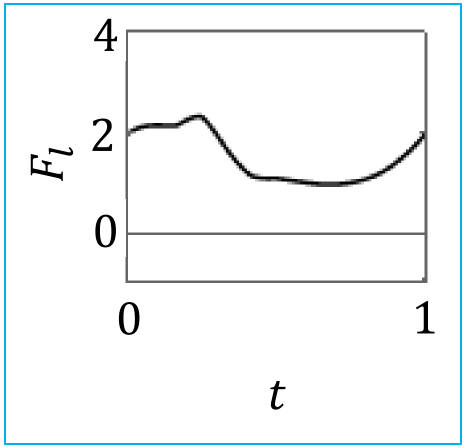
Arctic energy balance model

$$\frac{dE}{dt} = \big(1 - \alpha(E)\big)F_s(t) \;-\; \big(F_l(t) + BT(E,t)\big)$$

albedo | incoming solar radiation | outgoing longwave radiation





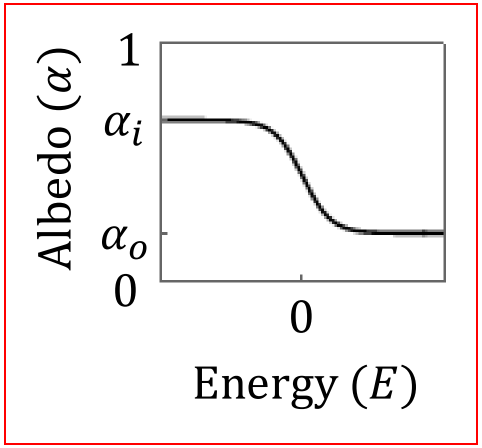


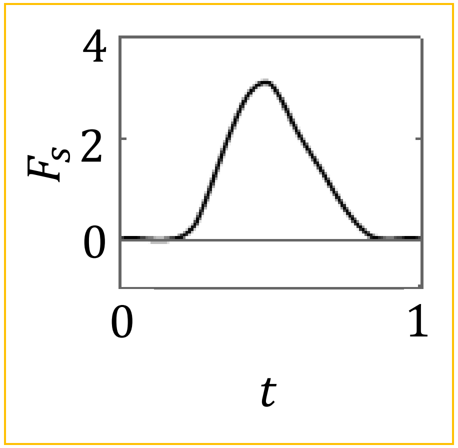
(Eisenman and Wettlaufer 2009)

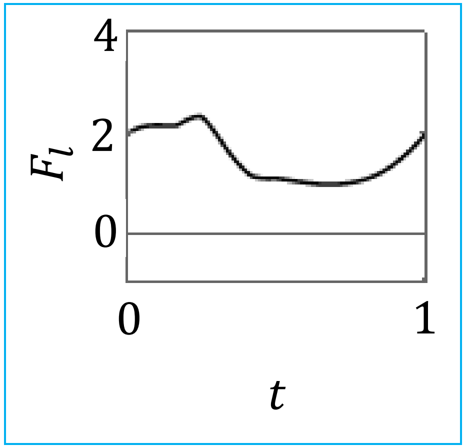
Arctic energy balance model

$$\frac{dE}{dt} = \big(1 - \alpha(E)\big)F_s(t) \; - \; \big(F_l(t) + BT(E,t)\big)$$

albedo | incoming solar radiation | outgoing longwave radiation | temperature



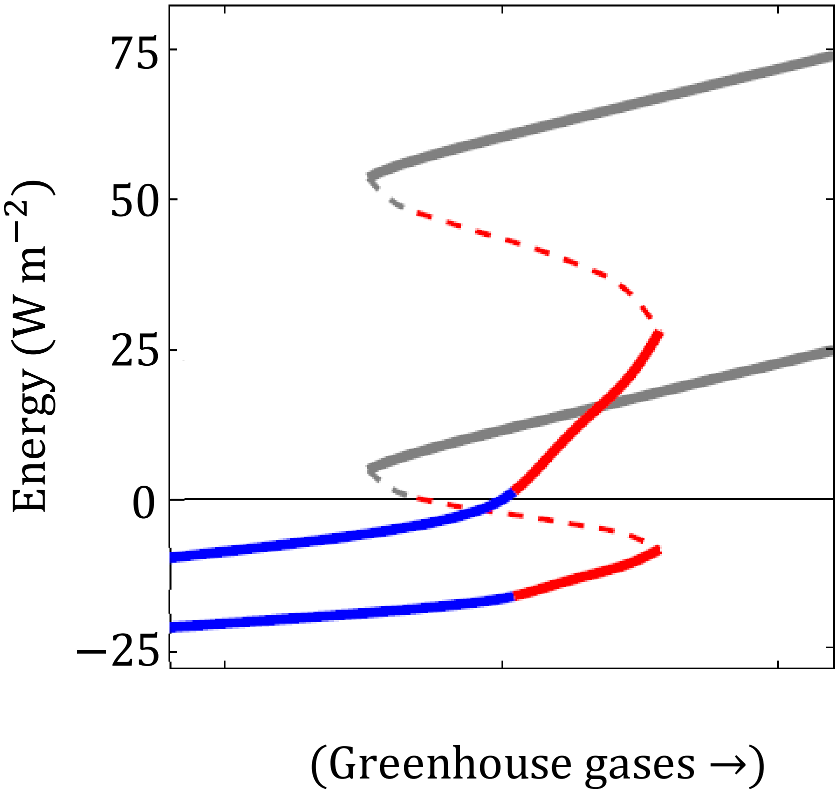




$$T \propto \begin{cases} E & \text{(ocean)} \\ 0 & \text{(melt)} \\ f(t,E) & \text{(ice)} \end{cases}$$

(Eisenman and Wettlaufer 2009)

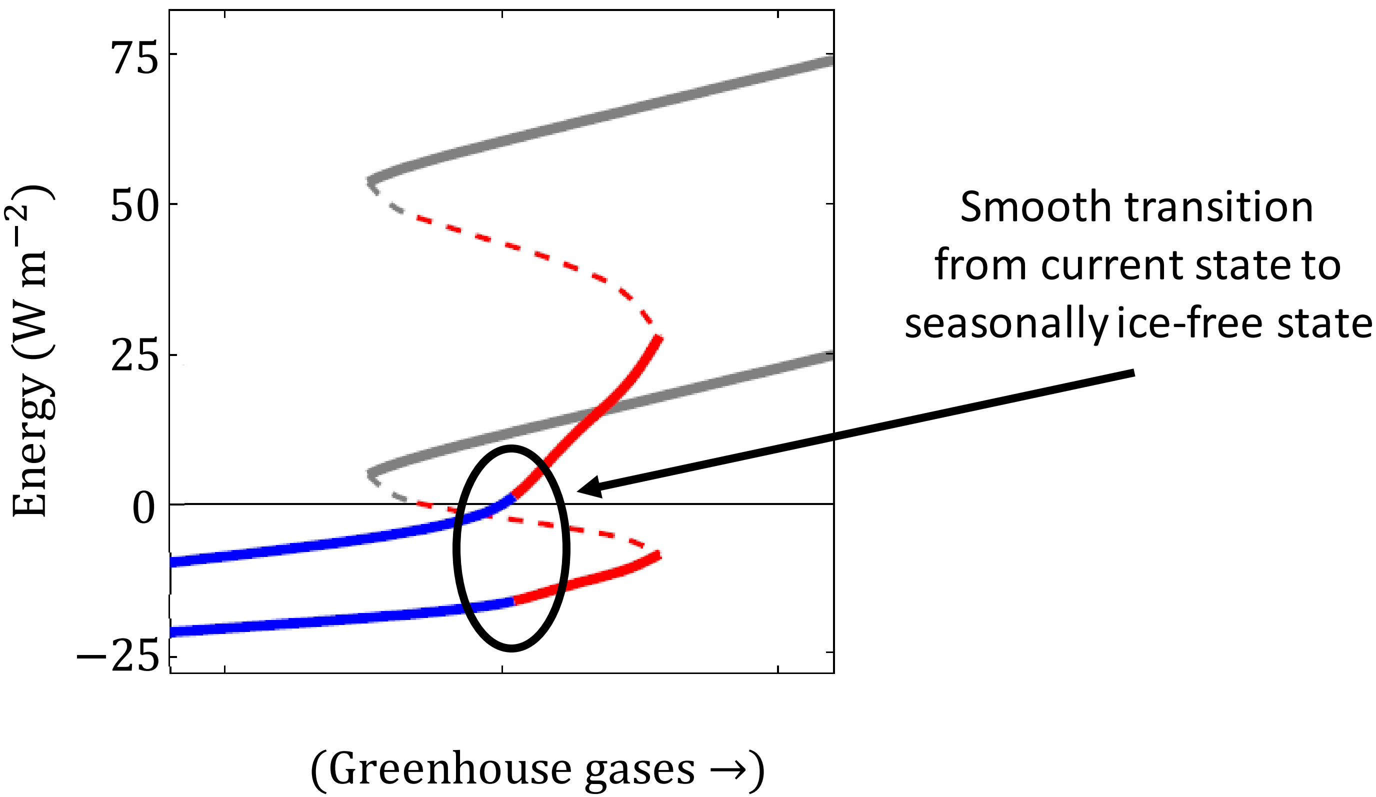
Bifurcation analysis: motivation



Perennially ice-free
Seasonally ice-free
Perennially ice-covered

(Eisenman and Wettlaufer 2009)

Bifurcation analysis: motivation



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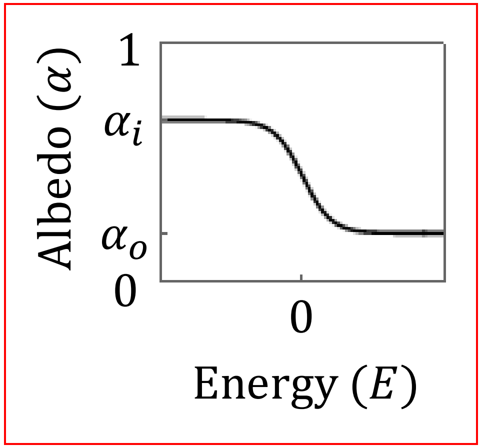
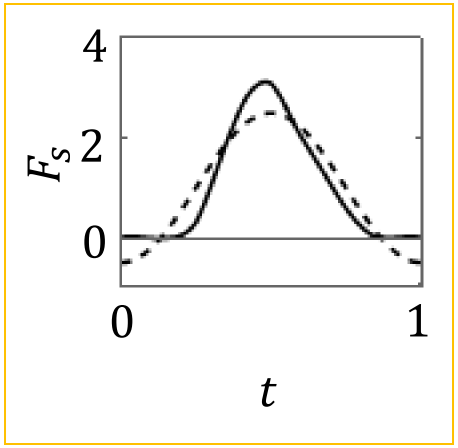
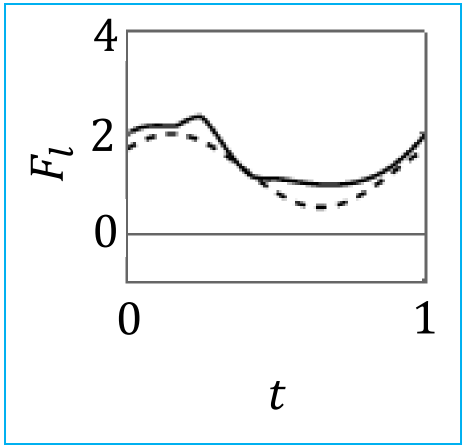




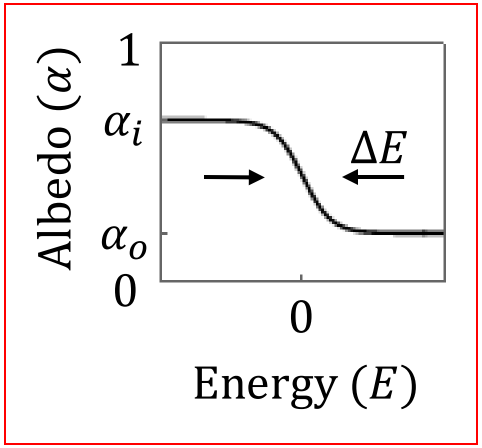


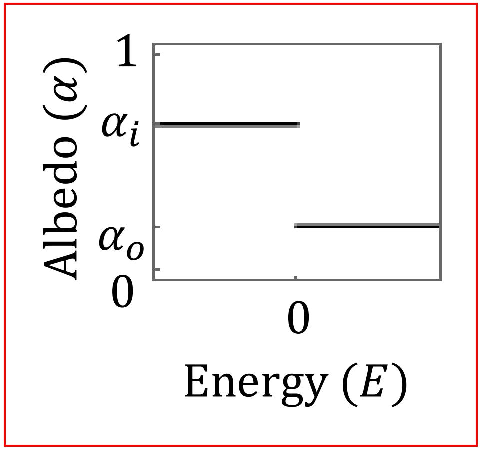
$$T \propto \begin{cases} E & \text{(ocean)} \\ 0 & \text{(melt)} \\ f(t,E) & \text{(ice)} \end{cases}$$

(Eisenman 2012)

Arctic energy balance model

$$\frac{dE}{dt} = \big(1 - \underbrace{\alpha(E)}_{\text{albedo}}\big)\underbrace{F_s(t)}_{\text{incoming solar radiation}} - \big(\underbrace{F_l(t)}_{\text{outgoing longwave radiation}} + B\underbrace{T(E,t)}_{\text{temperature}}\big)$$





Solutions in the piecewise-constant albedo limit

We can rewrite our model as:

$$\frac{dE}{dt} = \begin{cases} (1+\Delta\alpha)F_s(t) - \left(F_l(t) + BT(E,t)\right), & E > 0 \\ (1-\Delta\alpha)F_s(t) - \left(F_l(t) + BT(E,t)\right), & E < 0 \end{cases}$$

$$= \begin{cases} F_+(t) - BT(E,t), & E > 0 \\ F_-(t) - BT(E,t), & E < 0 \end{cases}$$

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$$= \begin{cases} F_+ - BE, & E \geq 0 \quad \text{[no ice]} \\ F_-, & E < 0, F_- > 0 \quad \text{[ice melting]} \\ \dfrac{\zeta F_-}{\zeta - E}, & E < 0, F_- < 0 \quad \text{[ice growing]} \end{cases}$$

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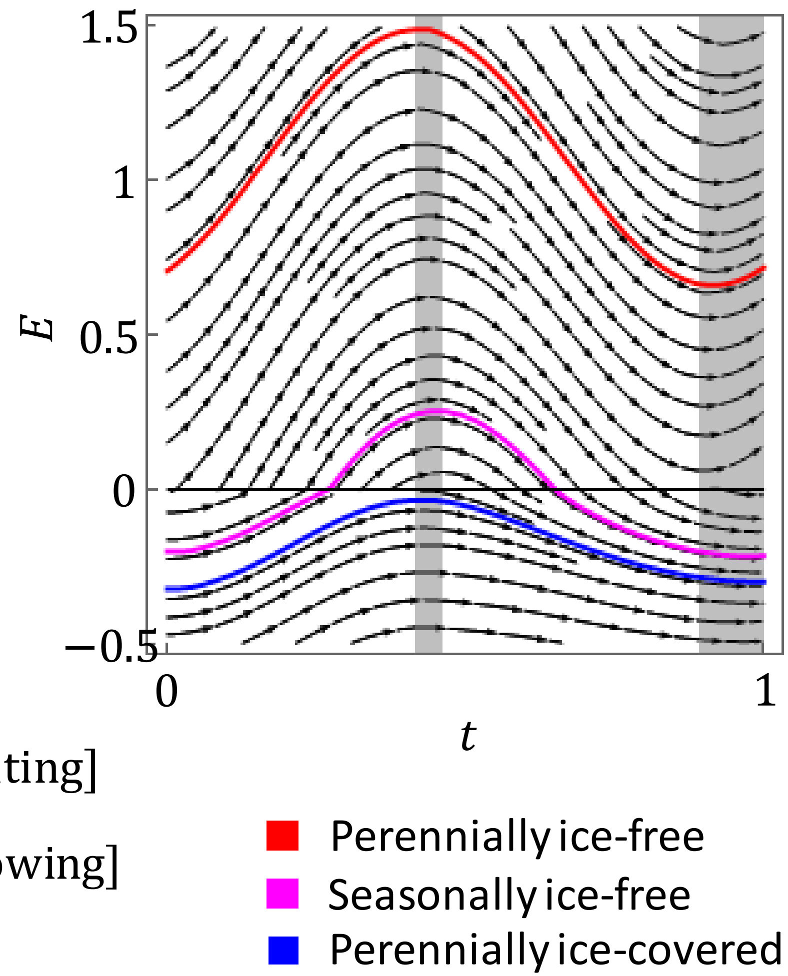
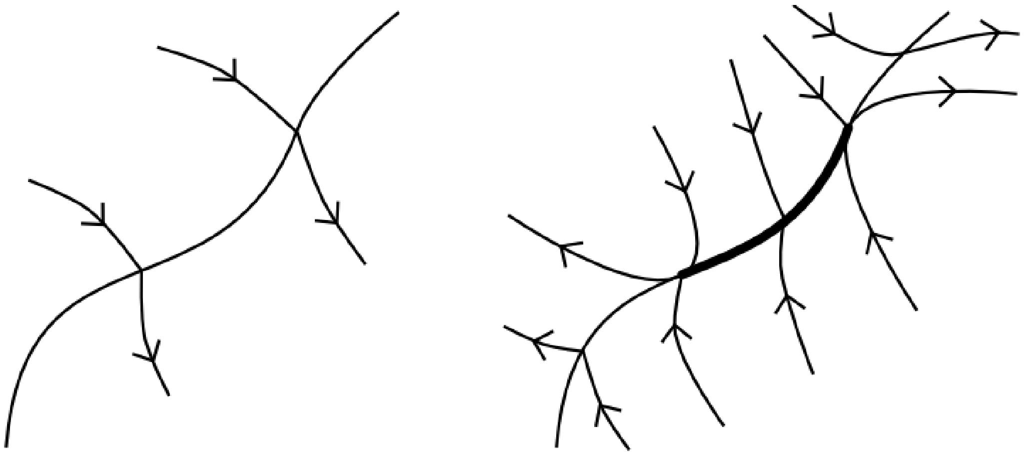


Piecewise-smooth dynamical systems

No sliding | Attracting Sliding | Repelling sliding



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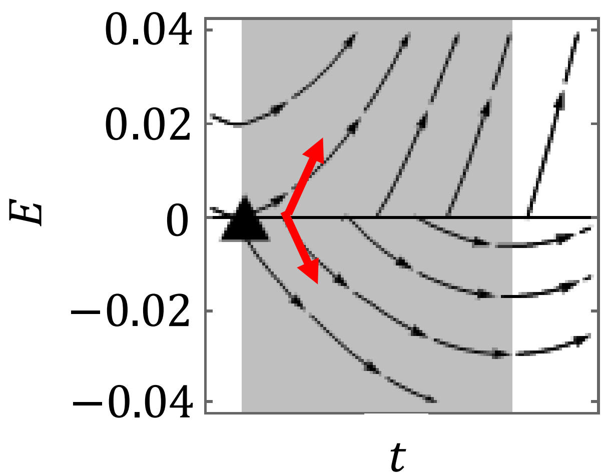


Repelling sliding intervals

Repelling sliding intervals

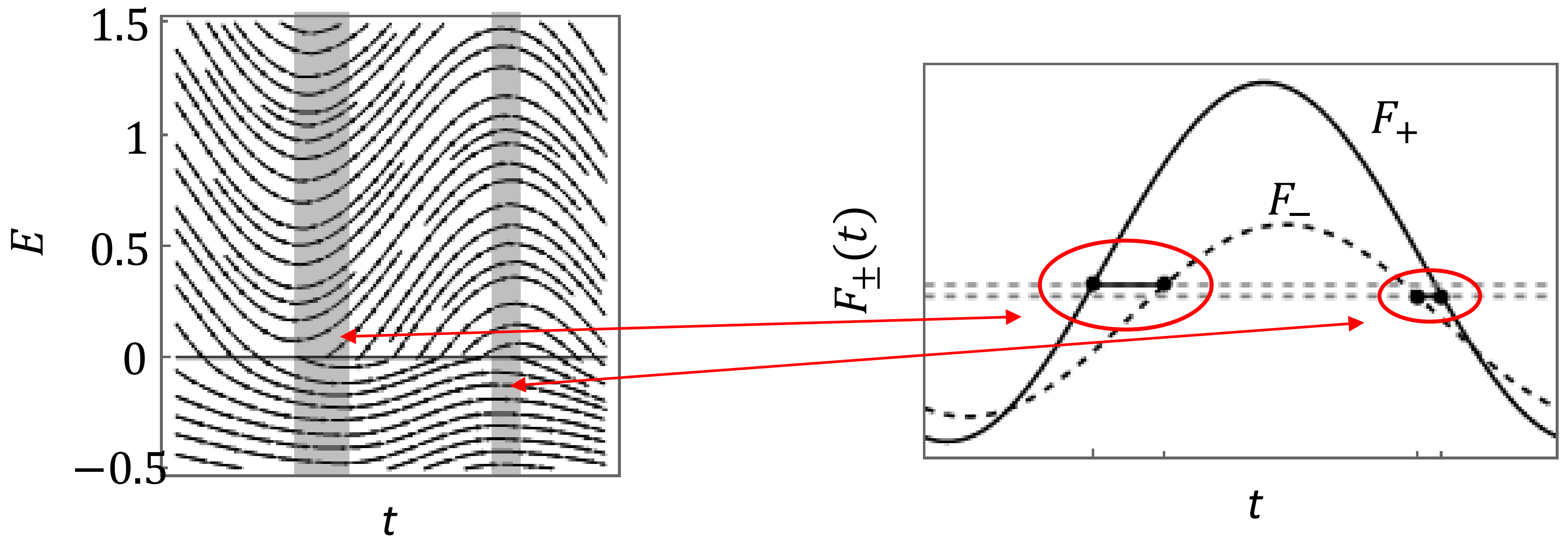
1. Width determined by $F_{\pm}$:

$$\frac{dE}{dt} = \begin{cases} F_{+}(t) - BT(E,t), & E > 0 \\ F_{-}(t) - BT(E,t), & E < 0 \end{cases}$$

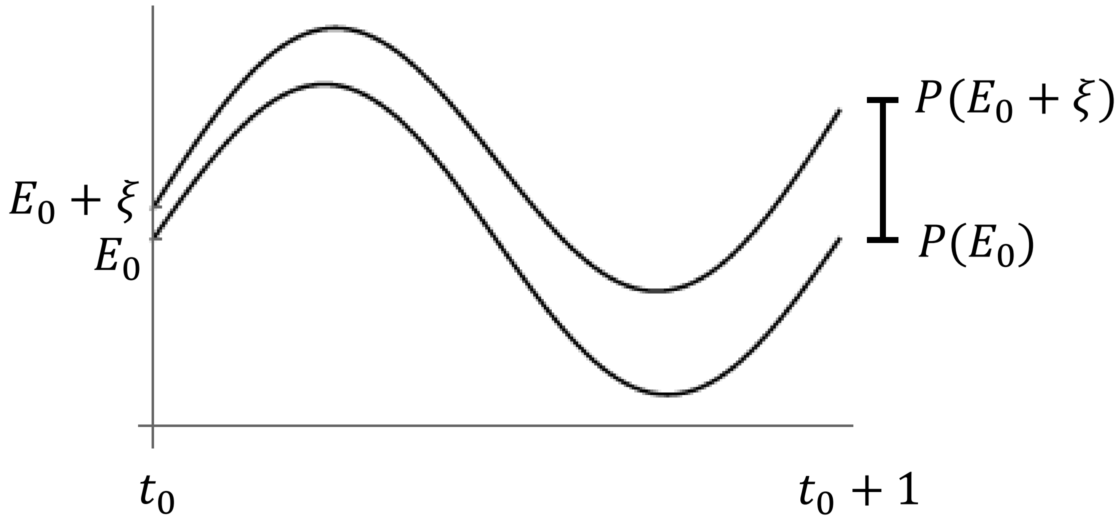
2. Affect stability of solutions
3. Introduce non-uniqueness

1. Sliding interval width

$$\frac{dE}{dt} = \begin{cases} F_+(t) - BT(E,t), & E > 0 \\ F_-(t) - BT(E,t), & E < 0 \end{cases}$$



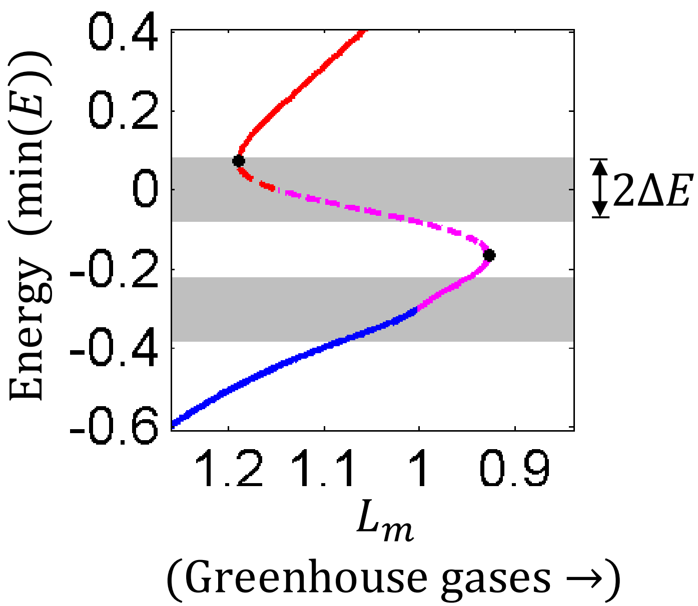
2. Stability: Floquet multiplier



$$\text{Floquet Multiplier} = \frac{P(E_0 + \xi) - P(E_0)}{\xi}$$

$$= \exp\left(\int_{t_0}^{t_0+1} \frac{\partial f}{\partial E}(\tau, E)\, d\tau\right)$$

2. Stability: Floquet multiplier



Perennially ice-free
Seasonally ice-free
Perennially ice-covered

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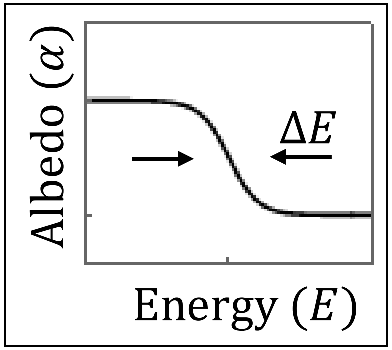

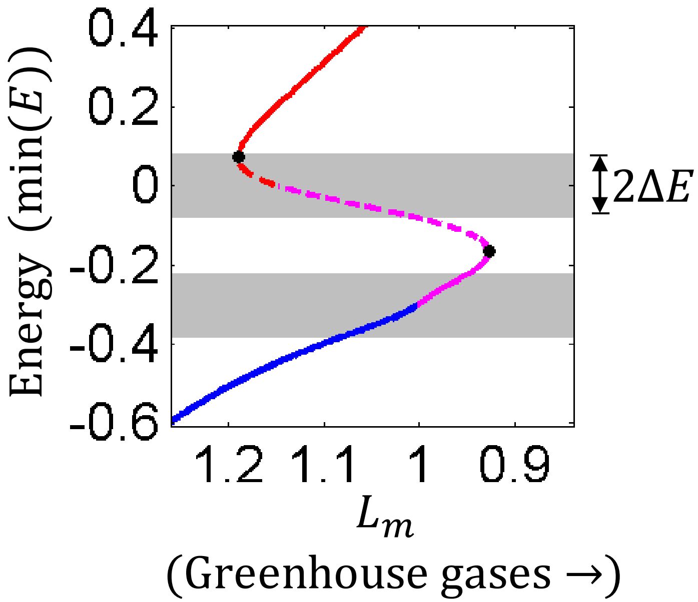

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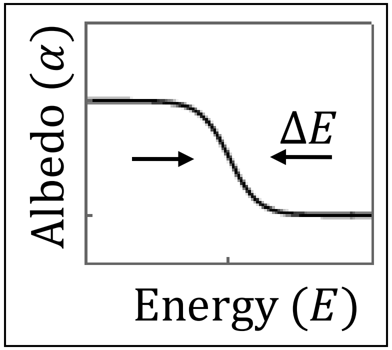

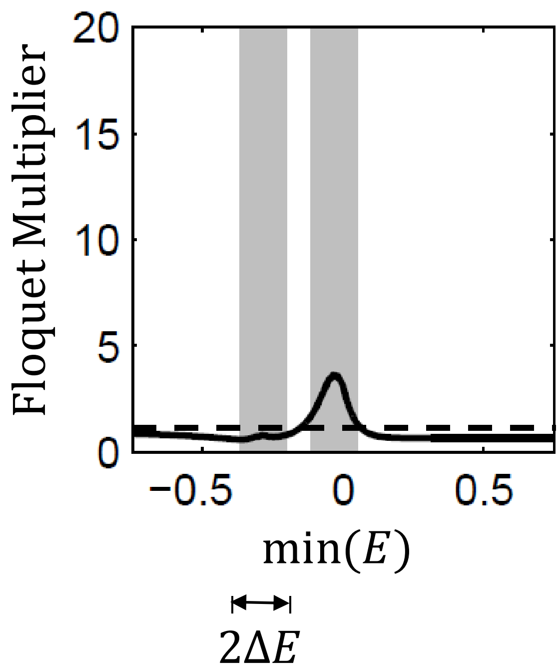

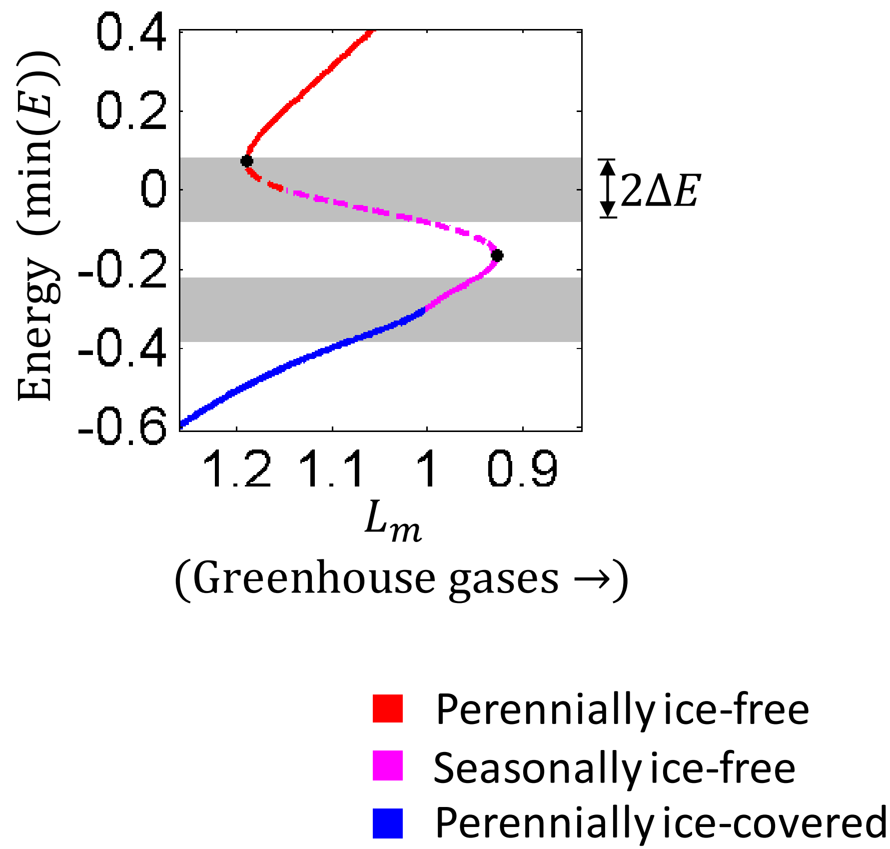

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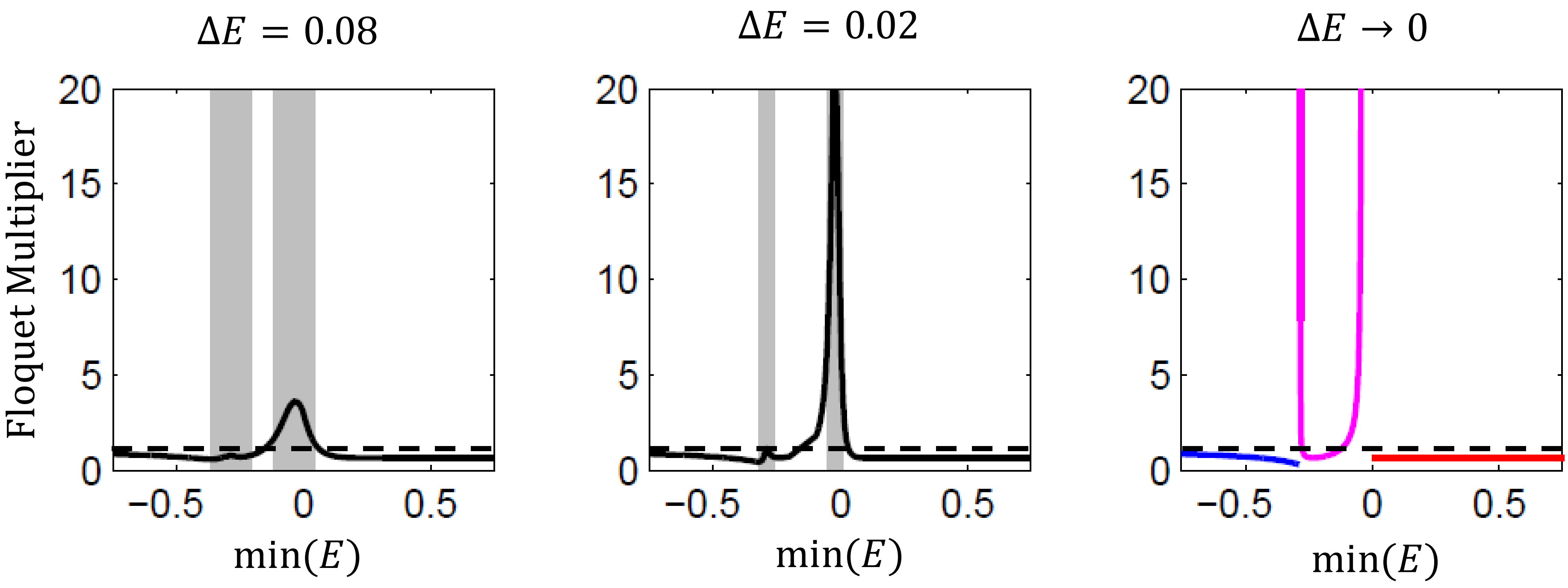

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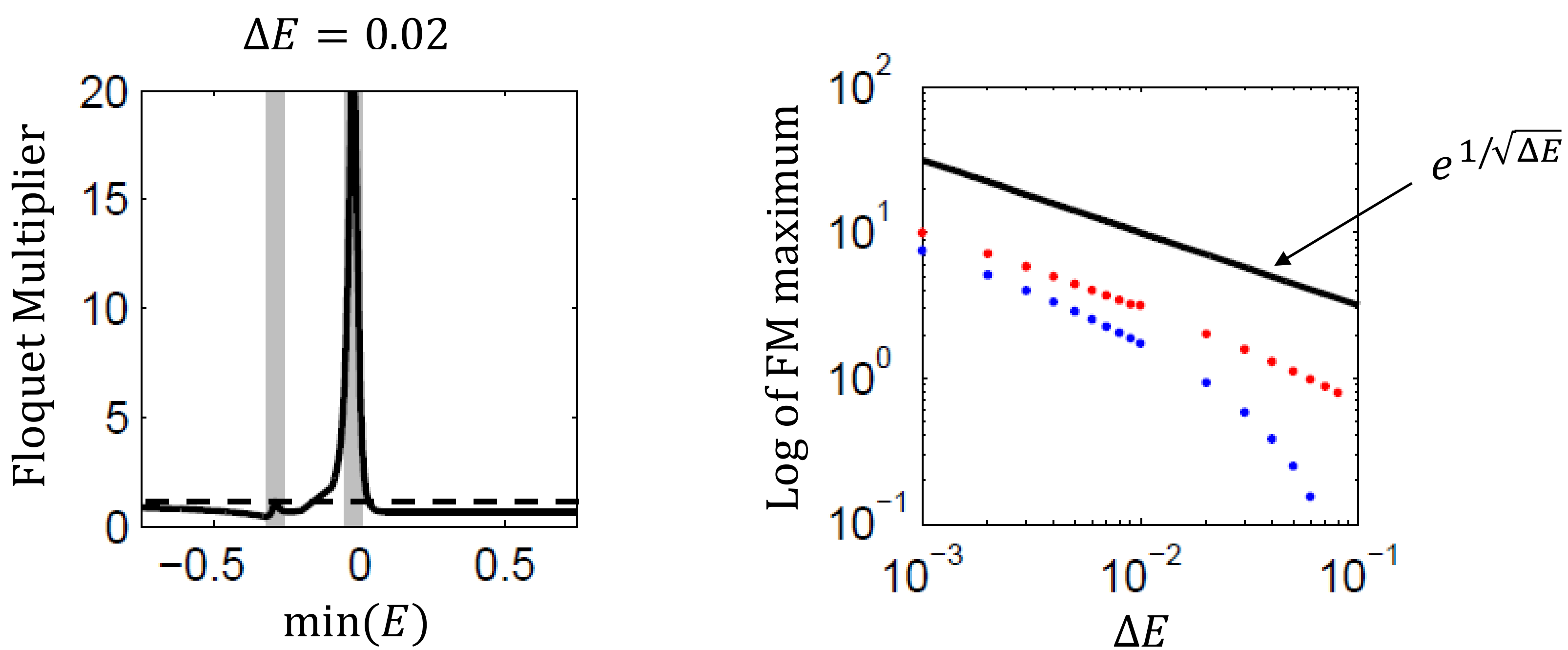


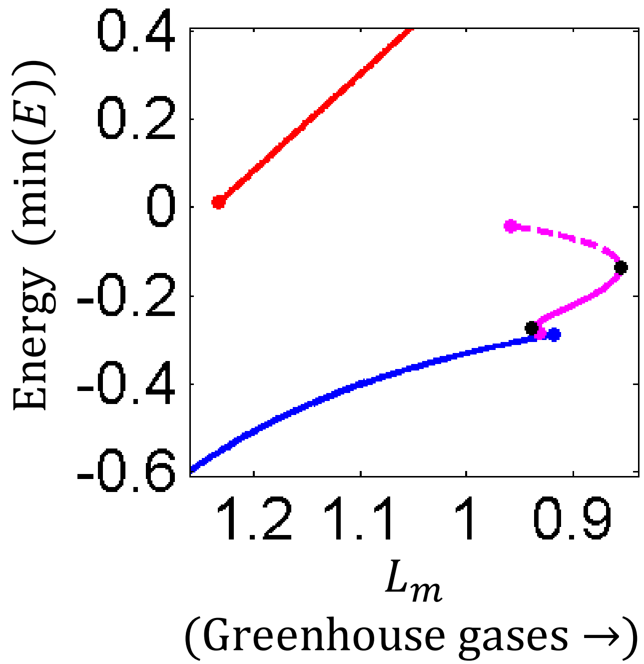
Maxima seem to increase like $e^{k/\sqrt{\Delta E}}$

Using Filippov solutions ($\Delta E = 0$) to approximate the Floquet multiplier:

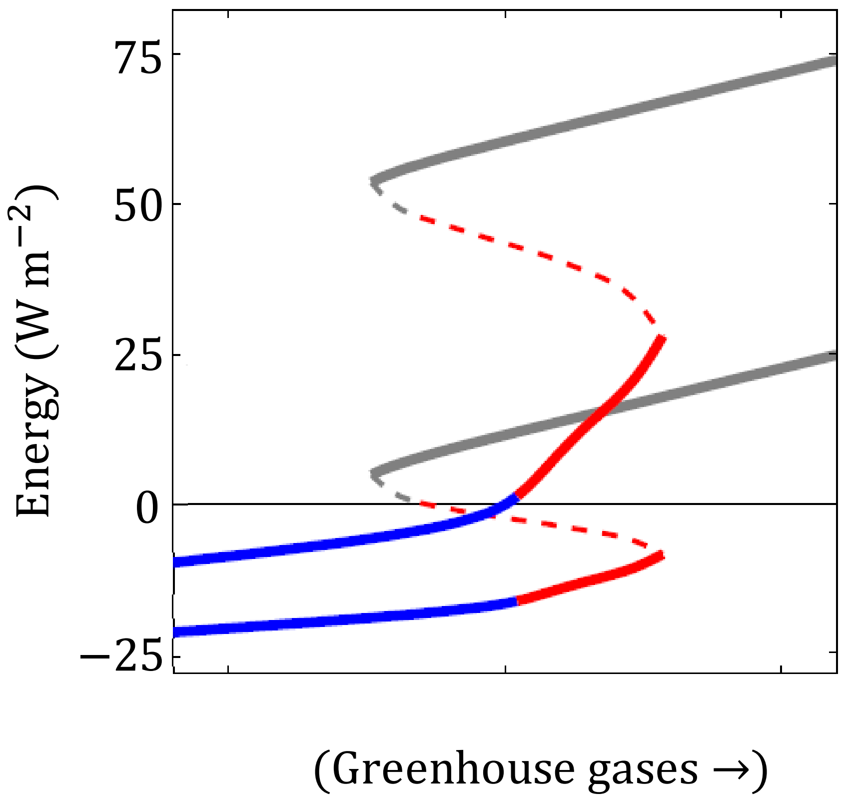
$$\exp\left(\int_{t_0}^{t_0+1} \frac{\partial f}{\partial E}(\tau, E)\, d\tau\right)$$

3. Non-unique solutions

- Omit non-unique periodic solutions from bifurcation diagram
- Leads to bifurcation diagram with 'gaps'
- Grazing-sliding bifurcations



Bifurcation analysis: motivation



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(Eisenman and Wettlaufer 2009)

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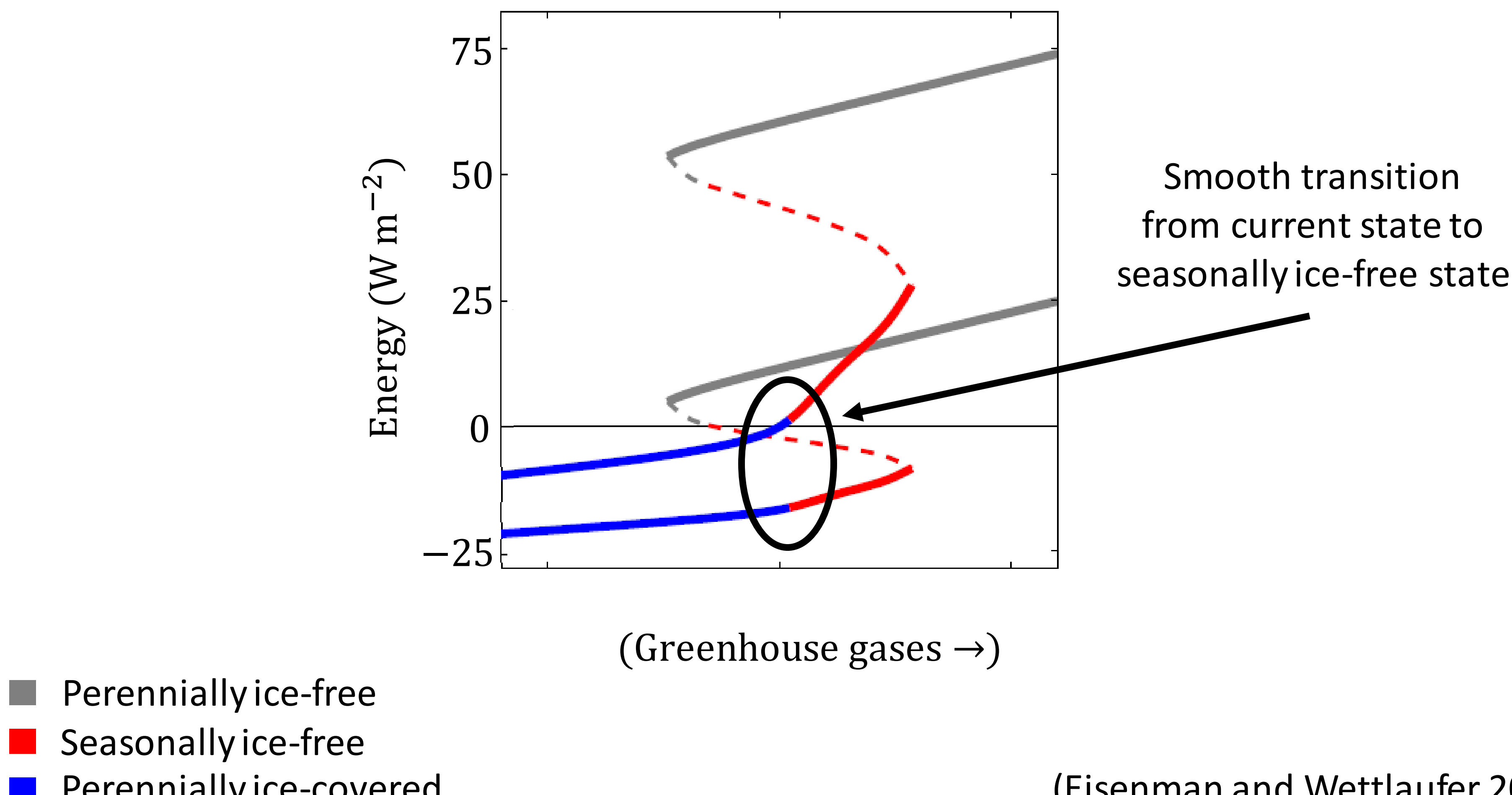
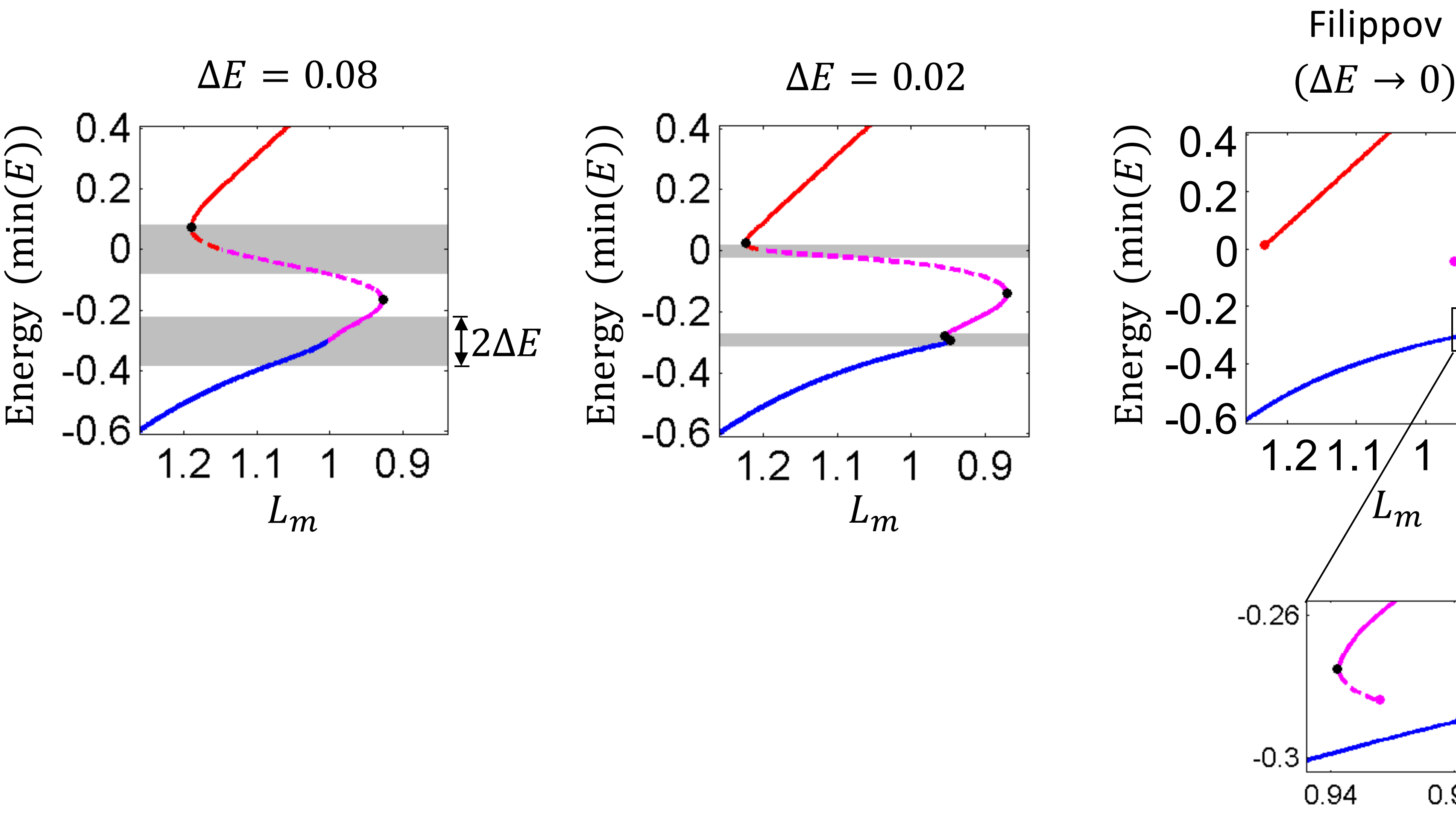


(Eisenman and Wettlaufer 2009)

Comparing piecewise-smooth to Filippov



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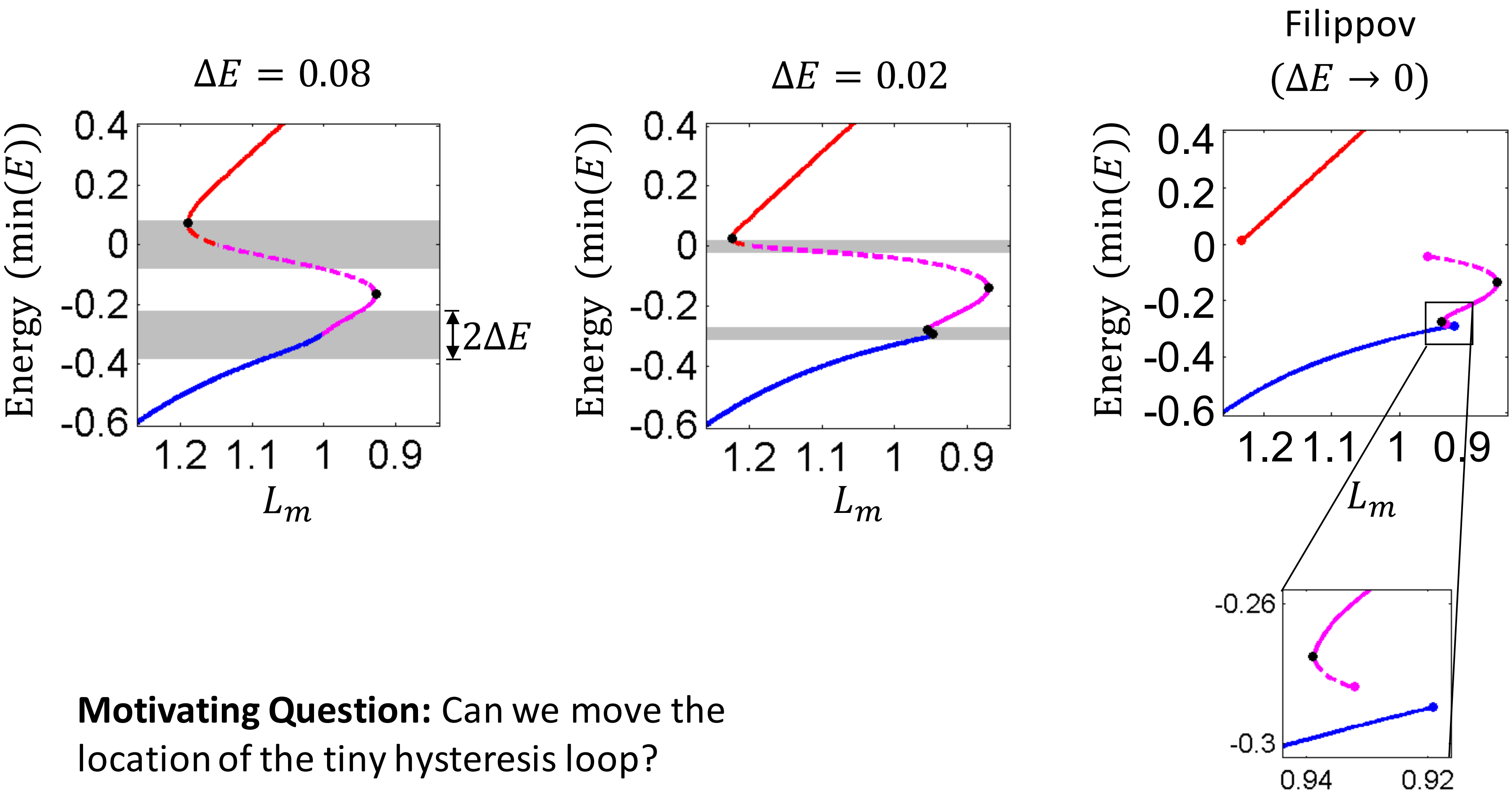
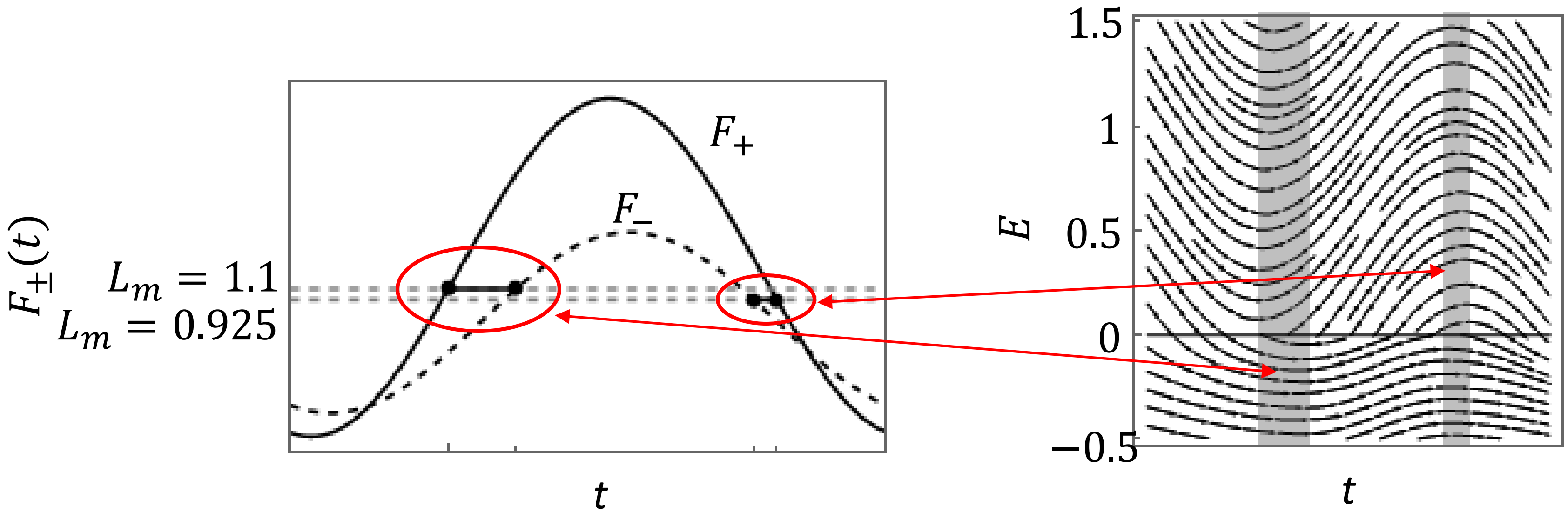


Motivating Question: Can we move the location of the tiny hysteresis loop?

Relationship between sliding intervals and gap size

Sliding width comes from $F_{\pm}$:

$$\frac{dE}{dt} = \begin{cases} F_{+}(t) - BT(E,t), & E > 0 \\ F_{-}(t) - BT(E,t), & E < 0 \end{cases}$$



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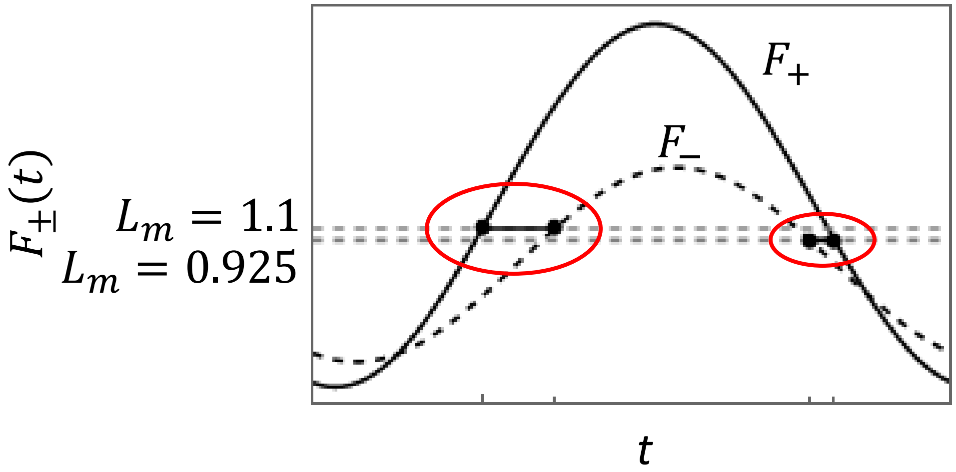
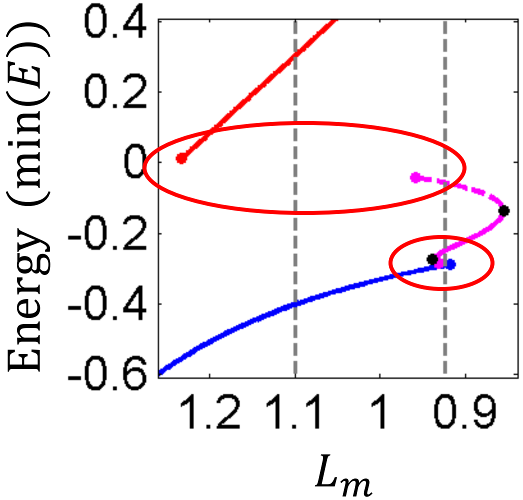




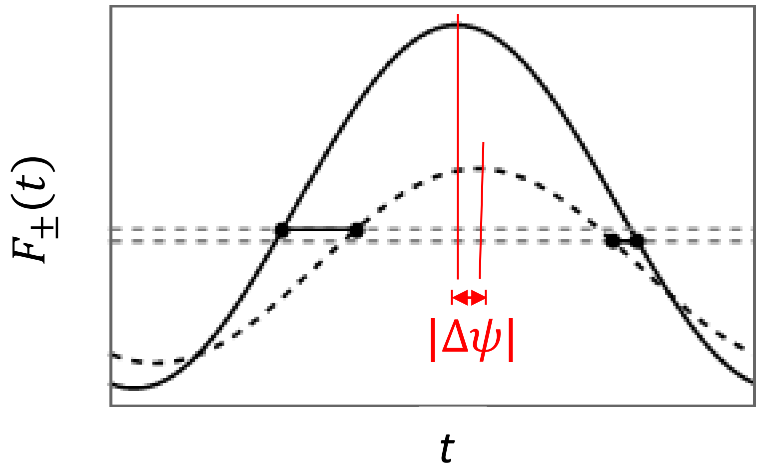
Relationship between sliding intervals and gap size

Map $F_{\pm}$ to a standard form:

$$F_{\pm} = (1 \pm \Delta_{\alpha})(1 - S_a \cos 2\pi t) - (L_m + L_a \cos 2\pi(t - \phi))$$

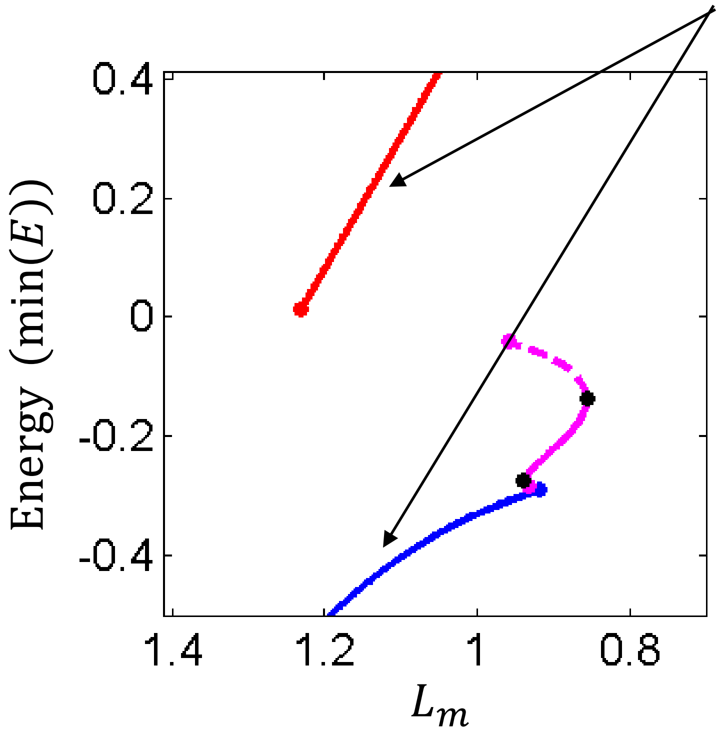
$$\longrightarrow \quad F_{\pm} = \overline{F_{\pm}} + \widetilde{F_{\pm}} \cos(2\pi t - \psi_{\pm})$$

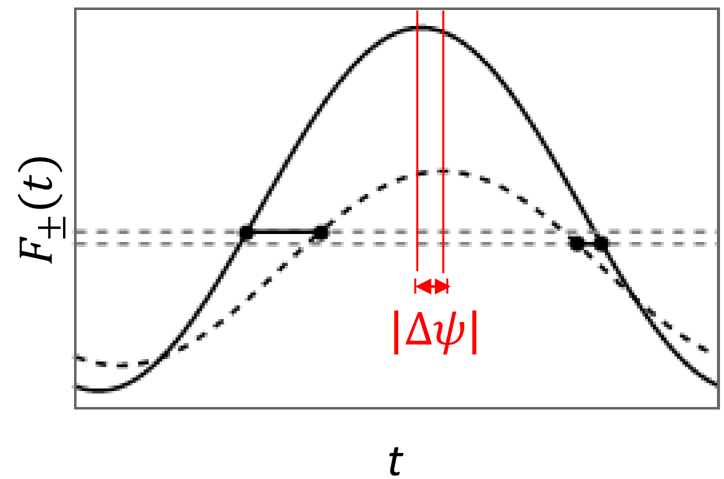
$$\Delta\psi = \psi_{+} - \psi_{-}$$



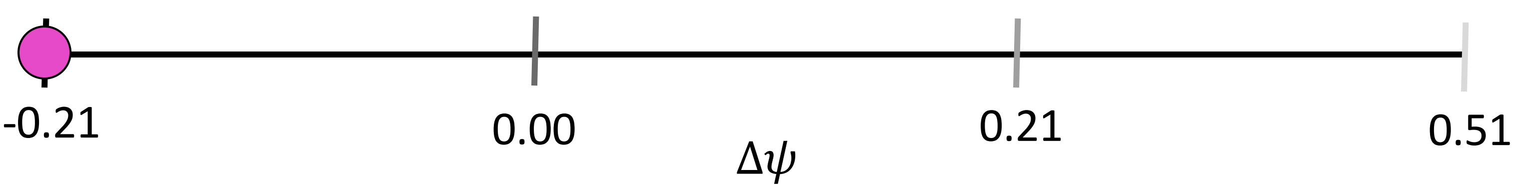
Varying $\Delta\psi$

don't change with $\Delta\psi$

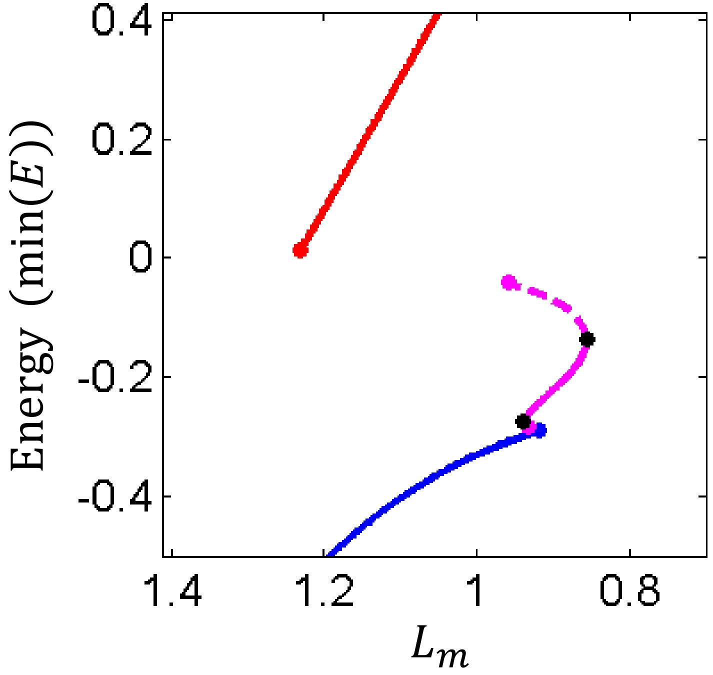


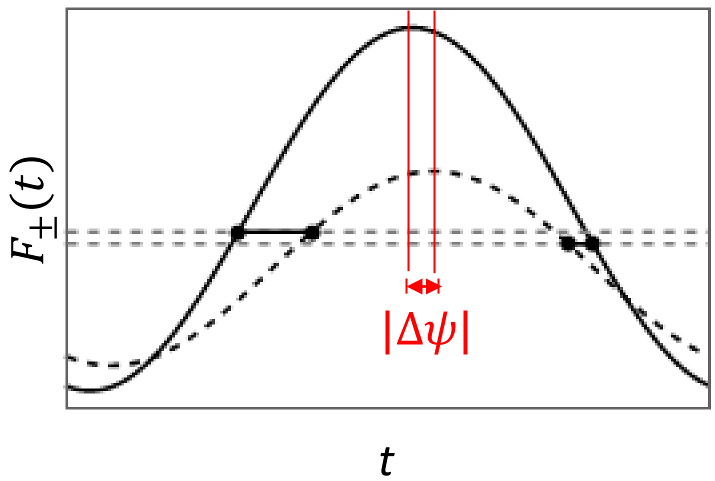


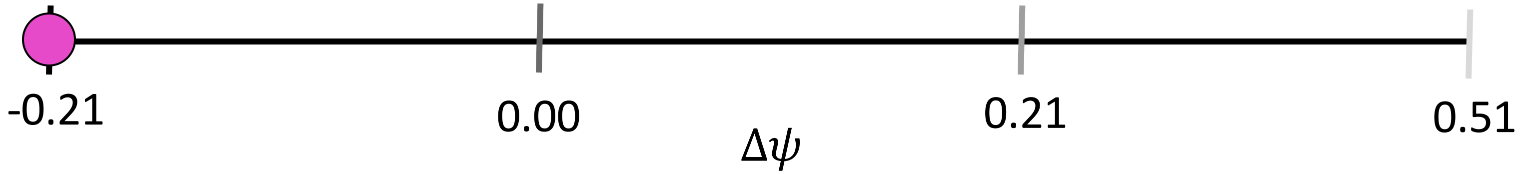
(Default value)



Varying $\Delta\psi$







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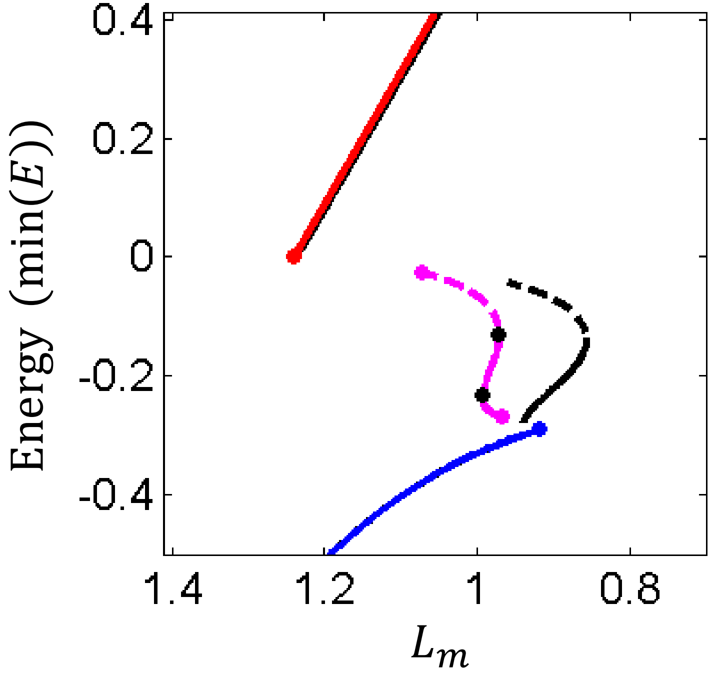
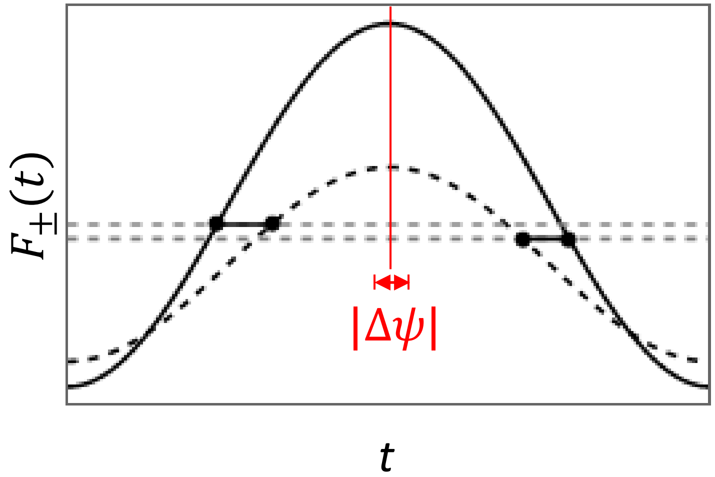
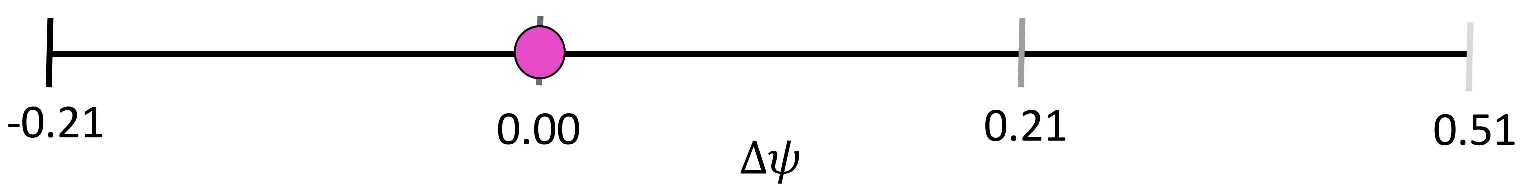
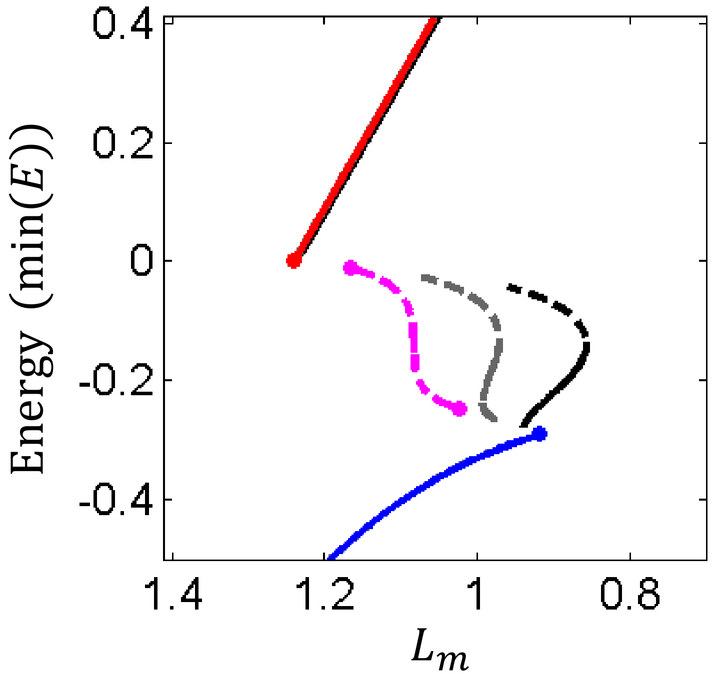
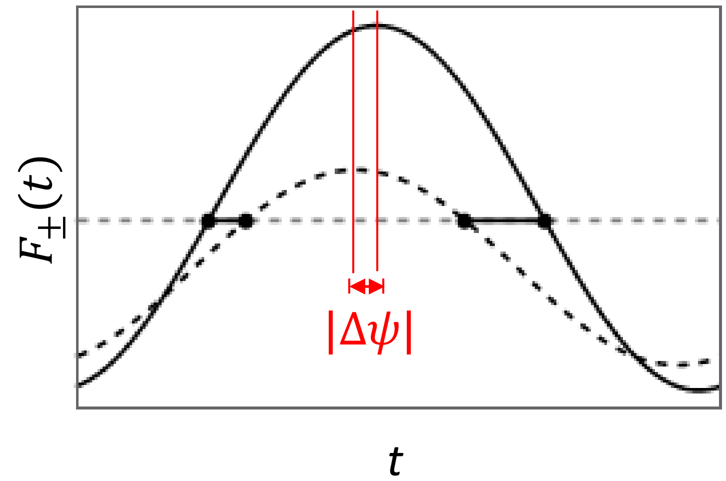


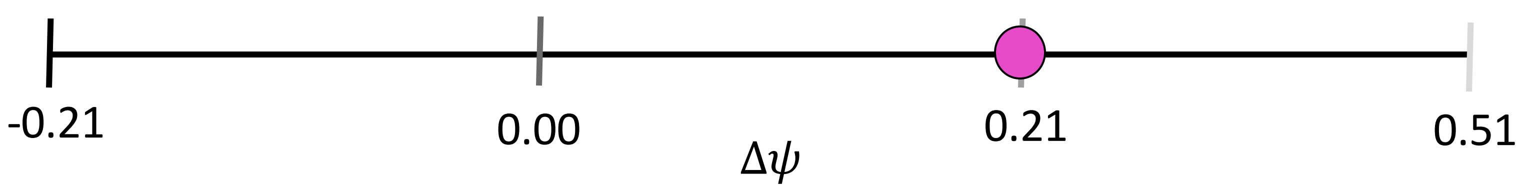




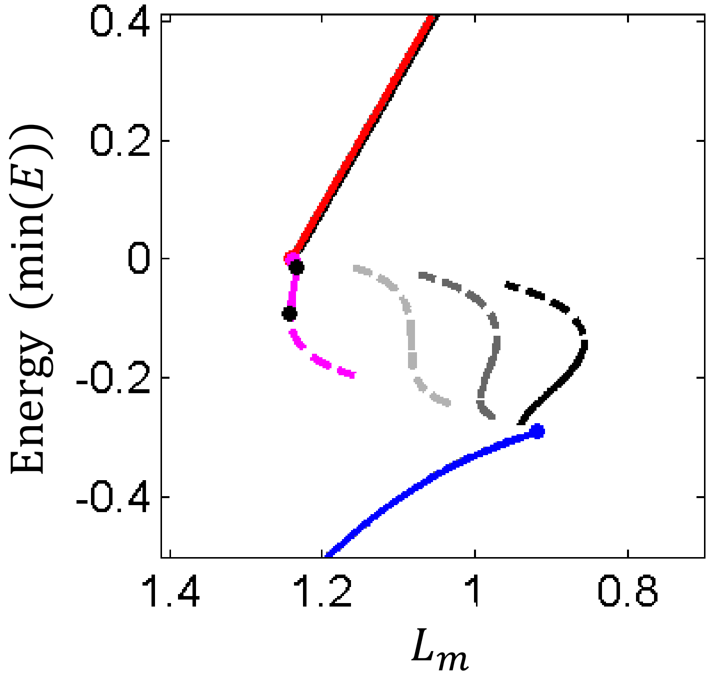
Varying $\Delta\psi$

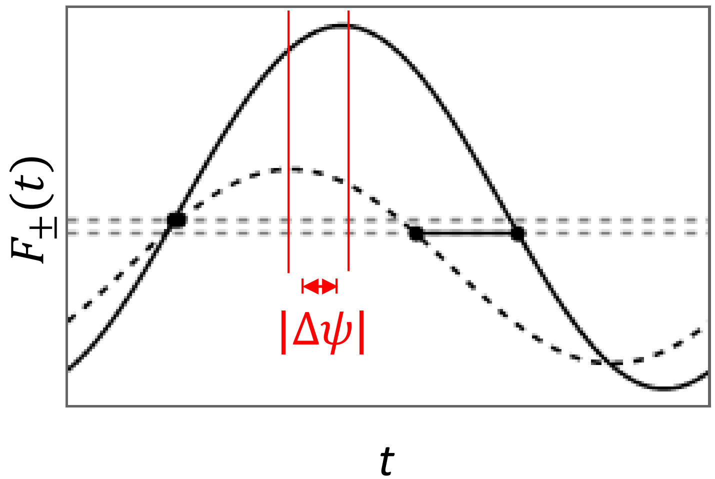


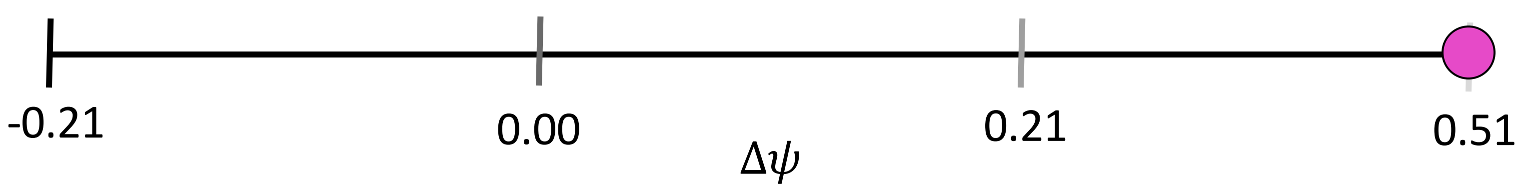


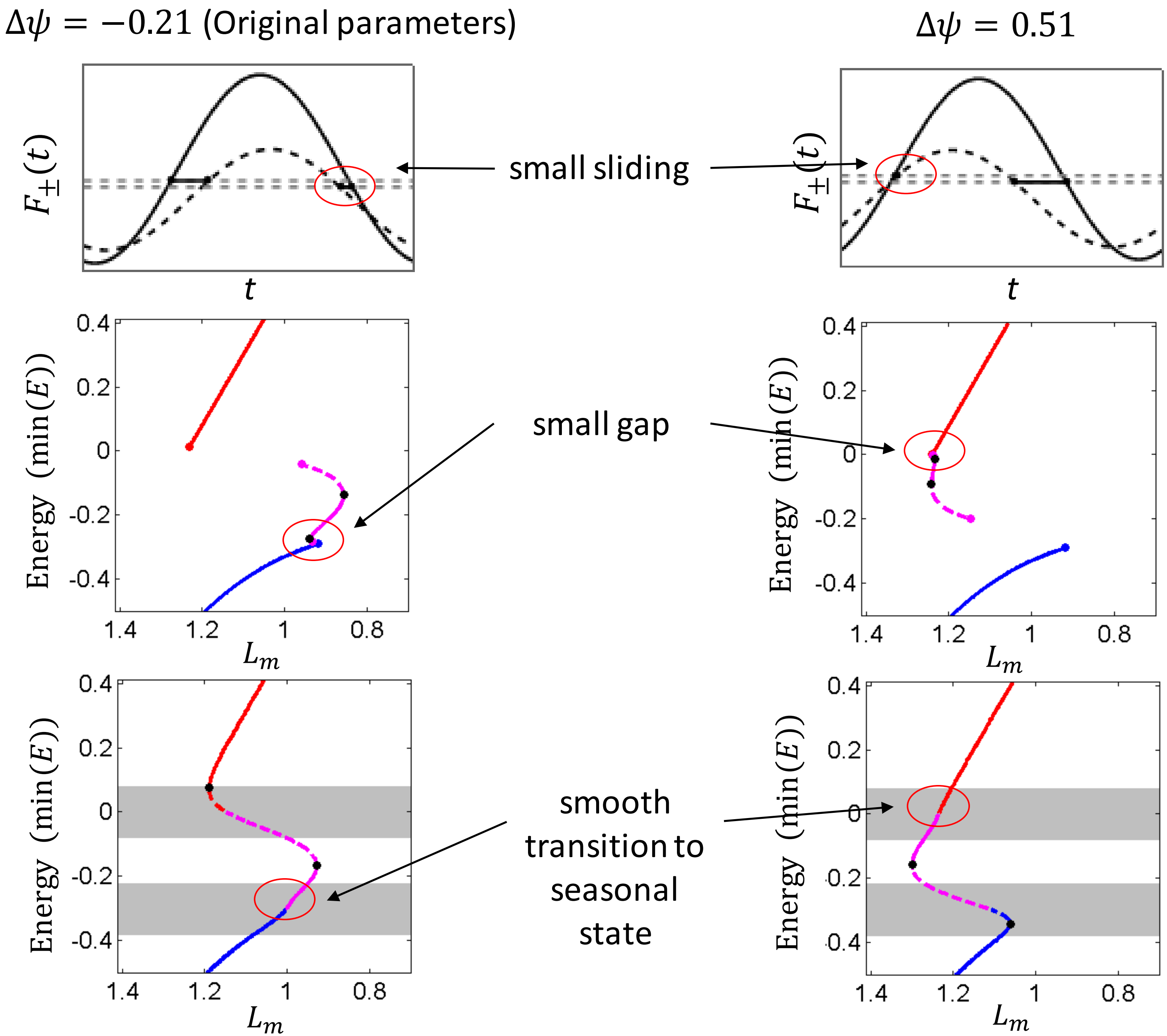


Varying $\Delta\psi$

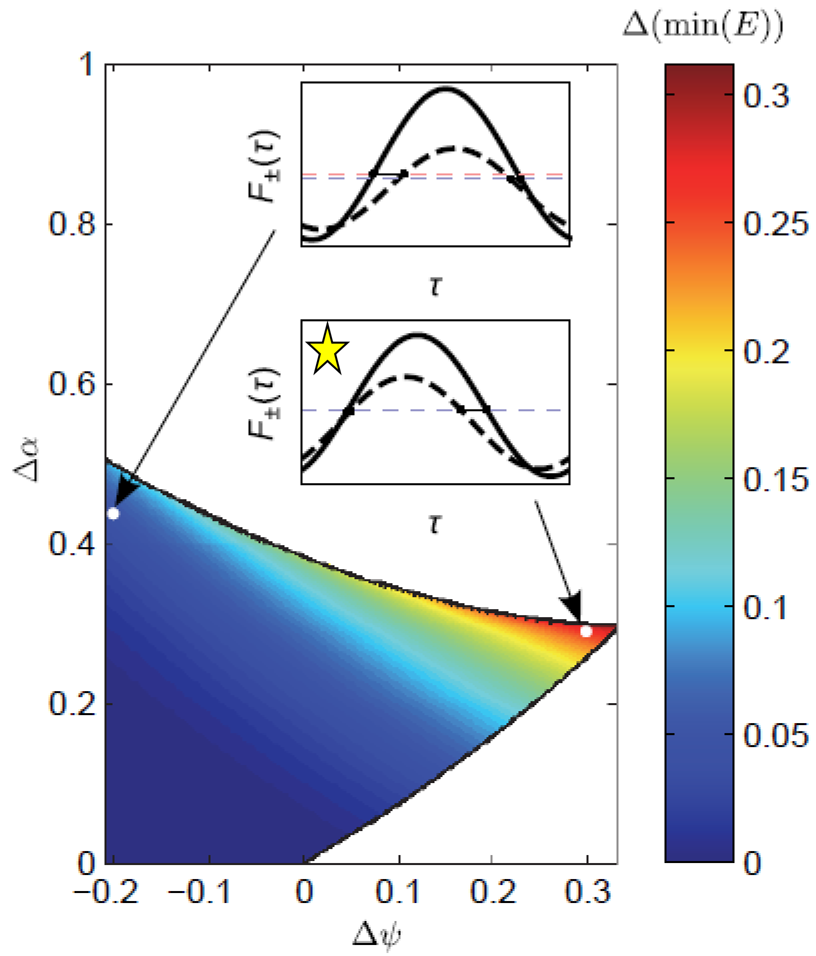


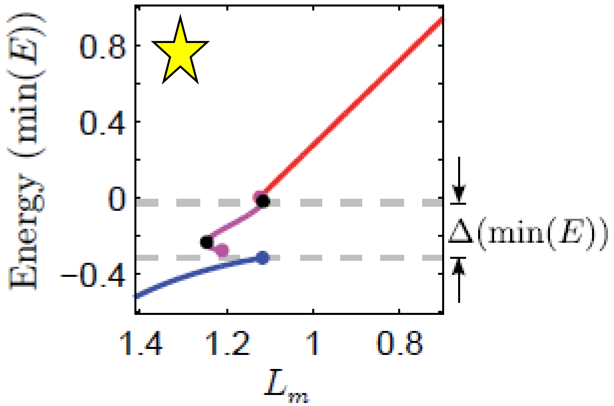


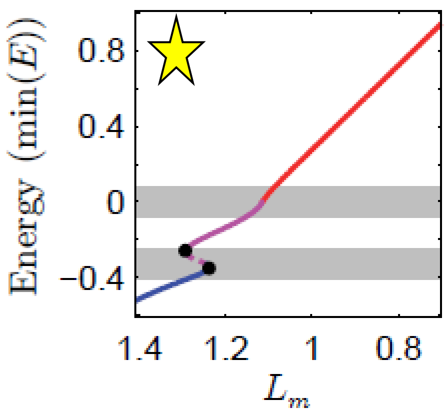



$\Delta\psi = -0.21$ (Original parameters)
$\Delta\psi = 0.51$
$F_{\pm}(t)$
t
small sliding
small gap
smooth transition to seasonal state
Energy $(\min(E))$
L_m
0.4
0.2
0
-0.2
-0.4
1.4
1.2
1
0.8

Size of the jump: $\Delta(\min(E))$

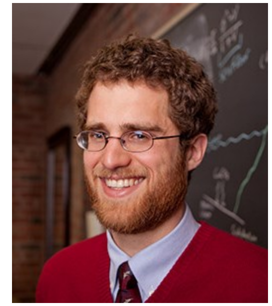






Thanks to:

Mary Silber
(Statistics, U Chicago)



Dorian Abbot
(Geophysical Sciences, U Chicago)

MCRN
Mathematics and Climate Research Network

NSF

Microsoft Research